بسم الله الرحمن الرحيم

SUMUDU DECOMPOSITION METHOD FOR FUZZY INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

Different results regarding different integro-differential equations, most of the time are not properly generalized, because of not satisfying some of the constraints. The field of fuzzy integrodifferential is very rich now a day because of its different applications and in the use of different physical phenomenon's. Solutions of FID are more generalized and have better applications. SDM is used for finding the solution of some linear and nonlinear FIDE. This method is easy to apply as well as it will give us more better results than other methods.

Acknowledgements

In the name of Almighty Lord, “All praise is due to Allah alone, the Sustainer of all the worlds. The Most Gracious, the Dispenser of Grace. Master of the Day of Judgment". Al-Fatiha - [1-3]. And most motivational phrase of holy Quran “Man gets whatever he strives for”. Surah Najm - [verse 39]

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Finally, words of appreciation to all my friends and family members who encourage me to achieve my goals.

ZAIN IQBAL

To my parents.

Introduction

A considerable lot of linear and nonlinear equations are a fundamental part in connected science and building fields. Nonlinear equations are seen in an alternate kind of physical problems, for example, liquid elements, plasma material science, strong mechanics, quantum field hypothesis, proliferation of shallow water waves, and numerous different models are controlled inside its area of legitimacy by incomplete differential equations. The wide utilization of these equations is the most imperative motivation behind why they have drawn mathematician's consideration. Regardless of this, they are difficult to discover an answer, either numerically or theoretically. Previously, dynamic examination endeavors were given a lot of regard for the investigation of getting exact or approximate solutions of this sort of equations. [1,2]

In the recent years the area of FIDEs has been developed a lot and have a key role in engineering. The elementary impression and arithmetic's of fuzzy sets were first introduced by L. A. Zadeh. Later, that the area of fuzzy derivative and fuzzy integration was studied, and some general results were developed. Fuzzy differential equations, FIE and FIDEs have much importance in the study of fuzzy theory and have much beneficial results for different problems. Modelling of different physical system under the differential way will give us different FIDEs [2,3] Also, FIDEs in fuzzy setting are a natural way to model ambiguity of dynamical systems. Consequently, different fields of sciences like Physics, Geographic, Medical and Biological Science pay much importance to the solution of different FIDEs. Solution of these equations can minimize different engineering problems. In Seikkala defined fuzzy derivatives while concept of integration of fuzzy functions was first introduced by Dubois and Prade. Alternative Analytic solution of FIDEs (nonlinear) type are often difficult to find. So, most of the time approximate solution is required. There are also useful numerical schemes that can produce a numerical approximation to solutions for some problems [6,7]

The literature on numerical solutions of IDE is large. We will use sumudu decomposition method for solving linear and nonlinear FIE. The method gives more realistic series solutions that converge very rapidly in physical problems. Sumudu Transform is also used for solving IDE which can be seen in [4,5]. IDE are transforms to FIDEs which are more general and gives better results. After applying Sumudu transform decomposition method is used for approximate solution. [8,9,10]

SUMUDU DECOMPOSITION METHOD FOR FUZZY INTEGRO-DIFFERENTIAL EQUATIONS

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DEPARTMENT OF MATHEMATICS

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Chapter NO 1

Basic Definitions Related to Fuzzy Integro-Differential Equations

Preliminaries

1.1 Integral Equation(IE):

The unknown function $\Upsilon(\xi)$ shows up under an integral symbol is known as integral equation. Usually we write an integral equation as follows [11].

$$\Upsilon(\xi) = f(\xi) + \int_{g(\xi)}^{h(\xi)} K(\xi, t)\Upsilon(t)dt \tag{1.1}$$

In the above Eq. (1.1) $k(\xi, t)$ and λ are the kernel and constant parameter respectively. Kernel is identified as function of dual variables ξ and t. $g(\xi)$ and $h(\xi)$ are recognized as the limitations for integration. The function $\Upsilon(\xi)$ that will be resolved shows up under integral symbol, it has a property that it will be looked in both the outside of integral symbol as well as inside the integral symbol. The functions that will be specified in progressive are $f(\xi)$ and $k(\xi, t)$. Limitations of integration can adopt both forms that is the variable, constant, or blended.

1.1.1 Types of IE.

IE show up in numerous kinds. Different sorts be contingent generally for the limitations of integration as well as the kernel of equality. In this content we will be worried on the accompanying sorts of IE. [11]

i. Fredholm IE
ii. Volterra IE
iii. Volterra-Fredholm IE
iv. Singular IE

1.1.2 Fredholm integral equation(FIE):

An equation in which the limitations for integration are constant or static as well as the unidentified function $\Upsilon(\xi)$ may show up just inside the integral sign is named FIE.

$$f(\xi) = \int_a^b K(\xi,t)\Upsilon(t)dt \qquad (1.2)$$

Eq. (1.2) is named as first kind of the FIE. For the second kind of FIE, the unidentified function $\Upsilon(\xi)$ looks in external side as well as internal side of the integral symbol, this will be explained well by the succeeding equation.

$$\Upsilon(\xi) = f(\xi) + \lambda \int_a^b K(\xi,t)\Upsilon(t)dt \qquad (1.3)$$

For instance,

$$\frac{sin\xi - \xi cos\xi}{\xi^2} = \int_0^1 \sin(\xi t)\,\Upsilon(t)dt$$

As well as,

$$\Upsilon(\xi) = (\xi + 3) + \frac{1}{2}\int_{-1}^{1} (\xi - t)\Upsilon(t)dt$$

Correspondingly.

1.1.3 Volterra Integral Equations(VIE)

There is a restriction for the VIE that at least single limit should be a variable. Likewise, FIE there are two varieties of VIE, that would be easier to described through,

$$f(\xi) = \int_0^\xi K(\xi,t)\Upsilon(t)dt \qquad (1.4)$$

Equality (1.4) is a VIE of 1st kind.

That is,

$$\xi e^{-\xi} = \int_0^\xi e^{t-\xi}\Upsilon(t)dt$$

$$\Upsilon(\xi) = f(\xi) + \lambda \int_0^{\xi} K(\xi,t)\Upsilon(t)dt \qquad (1.5)$$

Equality (1.5) is VIE of 2nd type,

For illustration,

$$\Upsilon(\xi) = 1 - \int_0^{\xi} \Upsilon(t)dt\ ,$$

1.1.4 Volterra-Fredholm Integral Equations

The VFIE stand up by different boundary value problems, like parabolic with the interaction with mathematical modelling for the spatiotemporal progress of a widespread, and after several physical and biological models. The VFIE equations seem in the works in two arrangements, namely

$$\Upsilon(\xi) = f(\xi) + \lambda_1 \int_0^{\xi} K_1(\xi,t)\Upsilon(t)dt + \lambda_2 \int_0^{\xi} K_2(\xi,t)\Upsilon(t)dt \qquad (1.6)$$

And,

$$\Upsilon(\xi,t) = f(\xi,t) + \lambda \int_0^{t}\int_{\Omega} F(\xi,t,\eta,\tau,\Upsilon(\eta,\tau))d\eta d\tau, (\xi,t)\epsilon\Omega \times [0,\mathrm{T}]$$

Where $f(\xi,t)$ and $F\big(\xi,t,\eta,\tau,\Upsilon(\eta,\tau)\big)d\eta d\tau$ represents analytic functions on $D = \Omega \times [0,\mathrm{T}]$ and Ω representing closed subset of $R^n, n = 1,2,3$ ... it is intriguing to take note of that (1.4) holds separate VIE and FIE, while (1.6) comprises mixed VIE and FIE. Additionally, the unknown functions $\Upsilon(\xi)$ and $\Upsilon(\xi,t)$ appears inner and outer the integral symbols. This is a trademark highlight of a second kind integral equation. On the off chance that the obscure function seems just inside the integral signs, the subsequent equations are of 1st type, be that as it may, won't be analyzed in this content. Examples for the both types are given below.

$$\Upsilon(\xi) = 2\xi + 4\xi^2 + 2 - \int_0^{\xi} \xi u(t)dt - \int_0^{1} t\Upsilon(t)dt \qquad (1.7)$$

$$\Upsilon(\xi,t) = \xi + t^3 + \frac{1}{5}t^2 - \frac{1}{4}t - \int_0^{t}\int_0^{1} (\tau - \eta)d\eta d\,\tau \qquad (1.8)$$

1.1.5 Singular Integral Equations:

An integral equation in which core converts into an infinite form at individual or both limit points is termed as singular integral equation.

In another way if at least one of the two limit points are infinite then the IE is also named as singular IE. For illustration,

$$\Upsilon(\xi) = f(\xi) + \lambda \int_{-\infty}^{\infty} e^{-|\xi - t|}\Upsilon(t)dt \tag{1.9}$$

As well as

$$f(\xi) = \int_{0}^{\xi} \frac{1}{(\xi - t)^{\alpha}}\Upsilon(t)dt, \qquad 0 < \alpha < 1 \tag{1.10}$$

1.1.6 Homogeneous and nonhomogeneous integral equation:

A 2nd order VIE or FIE is termed as homogeneous IE if $f(\xi)$ is similar as zero. Else it is named as inhomogeneous. It should be noted that this stuff is only for the 2nd type of equations.

For the clarification of this idea we have the equation as,

$$\Upsilon(\xi) = cos\xi + \int_{0}^{\xi} \xi t u(t)dt \tag{1.11}$$

$$\Upsilon(\xi) = \int_{0}^{\xi} (1 + \xi - t)\Upsilon^{5}(t)dt \tag{1.12}$$

Equation no (1.11) is nonhomogeneous and equation no (1.12) is homogeneous. Because in (1.10) $f(\xi) = cos\xi$ and in (1.11) $f(\xi) = 0$

1.1.7 Linear and nonlinear Integral equations:

An IE is called linear if the power of the unidentified function $\Upsilon(\xi)$ is one inside the integral symbol. If the power of unspecified function is other than one, or if the equality possesses the nonlinear terms for example, $e^u, Sinhu, Cosu, \ln(1+u)$ at that moment the IE is known as nonlinear. For the explanation of that idea we have the consideration as,

$$\Upsilon(\xi) = 2 - \int_0^{\xi} (\xi - t)\Upsilon(t)dt \tag{1.13}$$

$$\Upsilon(\xi) = 3 - \int_0^{1} (1 + \xi - t)\Upsilon^5(t)dt \tag{1.14}$$

Equation (1.13) is linear while the equation (1.14) is nonlinear.

1.2 Classification of Integro-differential Equations:

Various kinds of dynamical physical problems possess Integro-differential equations, specifically during the conversion of IVP's and BVP's. Differential operators as well as integral operators are involved in an Integro-differential equation. There could be any order for the presence of derivatives of the unknown function. In characterizing integro-differential equations, we will pursue a similar class utilized previously.

i. Fredholm Integrodifferential Equations
ii. Volterra Integrodifferential Equations
iii. Volterra-Fredholm Integrodifferential Equations

1.2.1 Fredholm Integrodifferential Equations:

Appearance of the Fredholm integro-differential equalities happen during the conversion of differential equation into integral equation. According to the definition of integro-differential equation of Fredholm kind presence of the unidentified function and its derivatives arise inside as well as outside the integral operator respectively. For FIE the limitations of integration are static that is constant. Presence of the differential as well as integral operatives make the equation a special kind of equation named integro-differential equation. To attain a particular solution the limits of integration should be available in FIDE. We have a FIDE as,

$$\Upsilon^n(\xi) = f(\xi) + \lambda \int_a^b K(\xi, t)\Upsilon dt \tag{1.15}$$

Where $\Upsilon^n(\xi)$ shows the nth order derivative of $\Upsilon(\xi)$. On the left half of the equation the derivatives of less order will appear with $\Upsilon^n(\xi)$. For example,

$$\Upsilon'(\xi) = 1 - \frac{1}{4}\xi + \int_0^1 \xi u(t)dt\,, \quad \Upsilon(0) = 0, \tag{1.16}$$

1.2.2 Volterra Integrodifferential Equation

Appearance of the Volterra integrodifferential equations happen during the conversion of IVP's into integral equation. According to the definition of integro-differential equation of Volterra kind presence of the unidentified function and its derivatives arise inside as well as outside the integral operator respectively. For VIE at minimum one of the limitations of integration are is variable. Presence of the differential as well as integral operatives make the equation a special kind of equation named integro-differential equation. To attain a solution the IC's of should be available in VIDE. We have a VIDE as,

$$\Upsilon^n(\xi) = f(\xi) + \lambda \int_0^\xi K(\xi, t)\Upsilon(t)dt \tag{1.17}$$

where Υ^n designates the derivative of order nth of $\Upsilon(\xi)$. Different derivatives of fewer order may show up by Υ^n at the left side. VIDE are specified as

$$\Upsilon'(\xi) = 3 + \frac{1}{4}\xi^2 - \xi e^\xi - \int_0^\xi t\Upsilon(t)dt\,, \quad \Upsilon(0) = 0, \tag{1.18}$$

1.2.3 Volterra-Fredholm Integrodifferential Equations(VFIE)

The VFIE emerge in indistinguishable way from Volterra-Fredholm integral equalities with at least one of conventional derivatives notwithstanding the integral operatives. The Volterra-Fredholm integrodiff erential equalities show up in the writing in two structures, to be specific

,

$$\Upsilon^n(\xi) = f(\xi) + \lambda_1 \int_0^\xi K_1(\xi, t)\Upsilon(t)dt + \lambda_2 \int_0^\xi K_2(\xi, t)\Upsilon(t)dt \tag{1.19}$$

And,

$$\Upsilon^n(\xi,t) = f(\xi,t) + \lambda \int_0^t \int_\Omega F(\xi,t,\eta,\tau,u(\eta,\tau))d\eta d\tau, (\eta,t)\epsilon\Omega \times [0,\mathrm{T}] \quad (1.20)$$

Where $f(\xi,t)$ and $F\big(\xi,t,\eta,\tau,u(\eta,\tau)\big)d\eta d\tau$ are investigative functions on $D = \Omega \times [0,\mathrm{T}]$ and Ω is a locked subset of $R^n, n = 1,2,3$ .... ,2,3. It will be fascinating to take memo of that (1.19) covers split VIE and FI equalities, though (1.20) contains blended integrals. Different derivatives of less request may show up too. In addition, the unfamiliar functions $\Upsilon(\xi)$ and $\Upsilon(\xi,t)$ shows up inner and outer side of the integral cyphers. This is a trademark highlight of a 2nd kind integral equality. On the off chance that the obscure functions seem just inner side the integral symbols, the subsequent equations are of 1st type. Preliminary conditions must be assumed to govern the results. IC's ought to be specified to decide the specific arrangement. Instances of the two kinds are specified as

$$\Upsilon'(\xi) = 2\xi + 3 - \int_0^\xi \xi\Upsilon(t)dt - \int_0^1 t\Upsilon(t)dt, \qquad \Upsilon(0) = 0$$

1.2.4 Homogeneous and nonhomogeneous integro-differential equations:

Classification of the 2nd type of integro-differential equations in homogeneous and inhomogeneous, if $f(\xi)$ in the 2nd type of Volterra or Fredholm IE's is indistinguishably zero, the equality is named homogeneous. else it is baptized as inhomogeneous. It is to be notified that this possession grasps only for 2nd type. To elucidate this idea, we think about the accompanying conditions,

$$\Upsilon^{(1)}(\xi) = 2 + \xi^4 - e^\xi + \int_0^\xi t\Upsilon(t)dt, \quad \Upsilon(0) = 0, \qquad (1.21)$$

$$\Upsilon^{(2)}(\xi) = \int_0^\xi \xi t\Upsilon(t)dt, \quad \Upsilon(0) = 1,\ \Upsilon^{(1)}(0) = 0 \qquad (1.22)$$

Equation (1.21) is in homogeneous because $f(\xi) = 2 + \frac{1}{3}\xi^2 - \xi e^\xi$ and equation (1.22) is homogeneous because $f(\xi) = 0$

1.2.5 Linear and nonlinear Integro-differential equations

If the obscure function inside the integral symbol possess a power one, the integro-differential equation is called linear. On the off chance if the power of function $\Upsilon(\xi)$ is greater from one, or if equation comprises nonlinear functions of $\Upsilon(\xi)$ for example, $e^u, Sinhu, Cosu, \ln(1+u)$ the integro-differential equality is called nonlinear. To clarify this idea, we think about the conditions,

$$\Upsilon'(\xi) = 1 + \int_0^1 \xi t e^t \Upsilon(t) dt, \quad \Upsilon(0) = 1, \tag{1.23}$$

$$\Upsilon'(\xi) = -1 + \frac{1}{6}\xi^2 - \xi e^{\xi} + \int_0^{\xi} t\Upsilon(t) dt, \quad \Upsilon(0) = 0, \tag{1.24}$$

Equation (1.23) is nonlinear because of nonlinear part under integral sign e^t. While (1.24) is linear because it contains linear part under integral.

1.3 Different method for solving integro-differential equations:

There are different methods for solving integro-differential equations. We will discuss some of methods which are often used for getting the solutions for integro-differential equations. [30]

i. Variational iteration technique(VIM)
ii. Series solution method
iii. Decomposition Method

1.3.1 Variational iteration method

The VIM was utilized to deal with Volterra fundamental conditions by changing over it to an IVP's or by changing over it to an equal integrodiff erential condition. The technique gives quickly concurrent progressive estimates of the correct arrangement if such a shut frame arrangement happens, and not segments as in (ADM).). The VIM grips linear as well as nonlinear issues in a similar way with no compelling reason to specific confinements, for example, the alleged Adomian polynomials that we requirement for issues which are not linear. The frame of nth order integro-differential equality is,

$$\Upsilon^n(\xi) = f(\xi) + \lambda \int_0^{\xi} K(\xi, t)\Upsilon(t) dt \tag{1.25}$$

Where $\Upsilon^n(\xi) = \frac{d^n\Upsilon}{d\xi^n}$ and $\Upsilon(0), \frac{d\,\Upsilon}{d\xi}(0), \frac{d^2\Upsilon}{d\xi^2}(0), \dots\dots\dots\dots \frac{d^{n-1}\Upsilon}{d\xi^{n-1}}(0)$ are initial conditions.

The integro-differential equality (1.25) as correctional functional is

$$\Upsilon_{k+1}(\xi) = \Upsilon_k(\xi) + \int_0^{\xi} \lambda(\eta)(\Upsilon_k^{(n)}(\eta) - f(\eta) - \int_0^{\eta} K(\eta, r)\tilde{\Upsilon}_k(r)dr)d\eta \qquad (1.26)$$

The VIM is utilized by putting on two fundamental advances. firstly it is required to decide the Lagrange multiplier λ which can be recognized ideally by means of integration by portions and by utilizing a inadequate diversity. Taking λ decided, an iterative formulation, without confined diversity, ought to be utilized for the assurance of the progressive approximation $\Upsilon_{k+1}(\xi), k \geq 0$ of the arrangement $\Upsilon(\xi)$. The zeroth guess $\Upsilon_0(\xi)$ can be any specific function. Be that as it may, the initial standards $\Upsilon(0), \Upsilon'(0), \dots$ are lean toward capably utilized for the particular zeroth estimation $\Upsilon_0(\xi)$ as will be perceived later. Subsequently, the solution is specified as,

$$\Upsilon(\xi) = \lim_{k\to\infty} \Upsilon_k\,(\xi)$$

It is helpful to shrink the Lagrange multipliers

$$\Upsilon^{(1)} + f\left(\Upsilon(\eta), \Upsilon^{(1)}(\eta)\right) = 0, \lambda = -1,$$

$$\Upsilon^{(2)} + f\left(\Upsilon(\eta), \Upsilon^{(1)}(\eta), \Upsilon^{(2)}(\eta)\right) = 0, \lambda = -1 = \eta - \xi$$

$$\Upsilon^{(3)} + f\left(\Upsilon(\eta), \Upsilon^{(1)}(\eta), \Upsilon^{(2)}(\eta), \Upsilon^{(3)}(\eta)\right) = 0, \lambda = -1 = -\frac{1}{2!}(\eta - \xi)^2$$

$$\Upsilon^{(2)} + f\left(\left(\Upsilon(\eta), \Upsilon^{(1)}(\eta), \Upsilon^{(2)}(\eta)\right), \dots\dots\dots \Upsilon^n(\eta)\right) = 0$$

$$\lambda = (-1)^n \frac{1}{(n-1)!}(\eta - \xi)^{n-1}$$

Variational iteration method

(for FIDE's)

The technique gives quickly united progressive approximations of the correct arrangement if such a shut shape arrangement exists, and not parts as in ADM. The VIM handles linear as well as nonlinear issues in a similar way with no compelling reason to specific limitations, for example, the supposed Adomian polynomials which are required for issues which are not linear. The frame of usual nth order integrodifferential equation is,

$$\Upsilon^{n}(\xi) = f(\xi) + \lambda \int_{a}^{b} K(\xi, t)\Upsilon(t)dt \tag{1.27}$$

The IC's are as $\Upsilon^{n}(\xi) = \frac{d^{n}\Upsilon}{dx^{n}}$ and $\Upsilon(0), \Upsilon^{(1)}(0) \ldots \ldots \ldots \ldots \Upsilon^{n-1}(0)$

The integrodifferential equation (1.27) as correction functional is,

$$\Upsilon_{k+1}(\xi) = \Upsilon_{k}(\xi) + \int_{0}^{\xi} \lambda(\eta)(\Upsilon_{k}^{(n)}(\eta) - f(\eta) - \int_{a}^{b} K(\eta, r)\tilde{\Upsilon}_{k}(r)dr)d\eta \tag{1.28}$$

The VIM is utilized by smearing both of fundamental advances. Firstly, we should need to decide the Lagrange multiplier λ which should be recognized ideally by means of integration and by utilizing a inadequate diversity. Taking λ decided, an iterative formulation, without confined diversity, ought to be utilized for the assurance of the progressive approximation $\Upsilon_{k+1}(\xi), k \geq 0$ of the arrangement $\Upsilon(\xi)$. The zeroth guess $\Upsilon_{0}(\xi)$ can be any specific function. Be that as it may, the initial standards $\Upsilon(0), \Upsilon'(0), \ldots$ are lean toward capably utilized for the particular zeroth estimation $\Upsilon_{0}(\xi)$ as will be perceived later. Subsequently, the solution is specified as,

$$\Upsilon(\xi) = \lim_{k\to\infty} \Upsilon_{k}(\xi)$$

1.3.2 Series solution method(SSM)

(For Volterra integro-differential equation)

We have another form in which we can define the function having analytical property "if a real function $\Upsilon(\xi)$ possess derivatives for all orders same like the Taylor series at any point b at its domain

$$\Upsilon(\xi) = \sum_{n=0}^{\infty} \frac{\Upsilon^{(n)}(b)}{n!}(\xi - b)^{n} \tag{1.29}$$

About b Eq. (1.29) shows convergence towards $\Upsilon(\xi)$. For easiness, the nonexclusive type of Taylor solution at $\xi = 0$ can be composed as

$$\Upsilon(\xi) = \sum_{n=0}^{\infty} a_n \xi^n \tag{1.30}$$

In this area we will utilize the application of SSM for fathoming VIDE of the 2nd kind. We have the expectations that the solution $\Upsilon(\xi)$ of the Voltera integro-differential equations is as,

$$\Upsilon^n(\xi) = f(\xi) + \lambda \int_0^{\xi} K(\xi, t)\Upsilon(t)dt, \qquad \Upsilon^{(k)}(0) = k!\, a_{k,} \quad 0 \le k \le (n-1) \tag{1.31}$$

is systematic, and accordingly has a Taylor arrangement of the frame agreed in (31), where the coefficients a_n will be resolved intermittently. The first few coefficients a_k can be controlled by utilizing the IC's as,

$$a_0 = \Upsilon(0), \qquad a_1 = \frac{d\Upsilon}{d\xi}(0), \qquad a_2 = \frac{1}{2!}\frac{d^2\Upsilon}{d\xi^2}(0), \qquad a_3 = \frac{1}{3!}\frac{d^3\Upsilon}{d\xi^3}(0)$$

etc. The lasting coefficients of a_k (1.30) will be dictated by the application of SSM to the VIDE (1.31). Substitution of (30) into the two sides of (30) supply

$$\left(\sum_{k=0}^{\infty} a_k \xi^k\right)^{(n)} = T\big(f(\xi)\big) + \int_0^{\xi} K(\xi, t)\left(\sum_{k=0}^{\infty} a_k t^k\right) dt \tag{1.32}$$

For simplicity we use

$$(a_0\xi + a_1\xi + a_2\xi^2 + \cdots)^n = T\big(f(\xi)\big) + \int_0^{\xi} K(\xi, t)(a_0 t + a_1 t + a_2 t^2 + \cdots)\, dt, \tag{1.33}$$

Whereas for $f(\xi)$ the Taylor arrangement is $T\big(f(\xi)\big)$. The integrodiff erential equality (1.31) will be changed over to a customary integral in (1.32) or (1.33) where as opposed to integration the obscure function $\Upsilon(\xi)$, terms of the structure $t^n, n \ge 0$,will be incorporated. Notice that since we are seeking for a result in series system, at that point on the off chance that $f(\xi)$ incorporates basic functions, for example, exponential functions, trigonometric functions, and so forth. Taylor extensions ought to be utilized for functions engaged with $f(\xi)$.

We first coordinate the side on the right of the integral symbol in (1.32) or (1.33) and gather the reflectance of alike exponents of ξ. Further we compare the reflectance of alike exponents of ξ into the both sides of the subsequent equality to decide a repeat connection in $a_{j,}\ j \ge 0$. Comprehending the repeat connection will prompt an entire resolve of the reflectance $a_{j,}\ j \ge 0$, where a portion of such reflectance will be utilized from the IC's. Taking decided the reflectance

$a_{j,}\ j \geq 0$, the solution in the form of series pursues promptly after replacing the determined coefficients into (1.30). The solution in the form of exact might be gotten if such a solution in exact form exist. In the event that a solution in exact form isn't realistic, the got arrangement will utilized for mathematical determinations. For this situation, the supplementary relations we assess, the advanced precision level we accomplish.

Series solution method (For Fredholm integro-differential equation)

Utilization of the SSM has been castoff earlier, and the general procedure of Taylor series for $\Upsilon(\xi)$ is as,

$$\Upsilon(\xi) = \sum_{n=0}^{\infty} a_n \xi^n \tag{1.34}$$

In this area we will utilize the application of SSM for fathoming FIDE of the 2nd kind. We have the expectations that the solution $\Upsilon(\xi)$ of the FIDE is as,

$$\Upsilon^k(\xi) = f(\xi) + \lambda \int_a^b K(\xi,t)\Upsilon(t)dt\,, \qquad \Upsilon^{(j)}(0) = a_{j,} \quad 0 \leq j \leq (k-1) \tag{1.35}$$

Is scientific, and in this manner has a Taylor arrangement of the structure given in (1.34), where the coefficients a_n will be resolved intermittently. Substitution of (34) into the two sides of (1.35) gives

$$\left(\sum_{n=0}^{\infty} a_k \xi^n\right)^{(k)} = T\big(f(\xi)\big) + \int_a^b K(\xi,t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt \tag{1.36}$$

For simplicity we use

$$(a_0\xi + a_1\xi + a_2\xi^2 + \cdots)^k = T\big(f(\xi)\big) + \lambda \int_a^b K(\xi,t)(a_0 t + a_1 t + a_2 t^2 + \cdots)\ dt, \tag{1.37}$$

Whereas for $f(\xi)$ the Taylor arrangement is $T\big(f(\xi)\big)$. The integrodiff erential equality (35) will be changed over to a customary integral in (1.36) or (1.37) where as opposed to integration the obscure function $\Upsilon(\xi)$, terms of the structure $t^n, n \geq 0$,will be incorporated. Notice that since we are seeking for a result in series method, at that point on the off chance that $f(\xi)$ incorporates basic functions, for example, exponential functions, trigonometric functions, and so forth. Taylor extensions ought to be utilized for functions engaged with $f(\xi)$.

We first coordinate the side on the right of the integral symbol in (1.36) or (1.37) and gather the reflectance of alike exponents of ξ. Further we compare the reflectance of alike exponents of x into the both aspects of the subsequent equality to decide a repeat connection in $a_{j,}\ j \geq 0$. Comprehending the repeat connection will prompt an entire purpose of the reflectance $a_{j,}\ j \geq 0$, where a portion of these coefficients will be utilized from the IC's. Taking decided the coefficients $a_{j,}\ j \geq 0$, the solution in the form of series pursues promptly after replacing the determined coefficients into (1.35). The solution in the form of exact might be gotten if such a solution in exact form exist. In the event that a solution in exact form isn't realistic, the got arrangement have an option of utilizing for mathematical determinations. For this situation, the supplementary relations we assess, the advanced precision level we accomplish.

FIDE gives solution in the exact frame if the arrangement $\Upsilon(\xi)$ be in the form of polynomial. Be that as it may, if the arrangement is some other straightforward function, for example, $sin\xi, e^{\xi}$ and so forth, the SSM stretches the solution in exact form to adjusting few of the coefficients $a_j j \geq 0$.

1.3.3 Direct decomposition method

(for Fredholm integro-differential equation)

$$K(\xi,t) = g(\xi)h(t) \tag{1.38}$$

The standard form of the FIDE is as

$$\Upsilon^n(\xi) = f(\xi) + \int_a^b K(\xi,t)\Upsilon(t)dt\,, \qquad \Upsilon^{(k)}(0) = b_{k,} \quad 0 \leq k \leq (n-1) \tag{1.39}$$

Where $u^n(\xi)$ specifies the nth derivative of $\Upsilon(\xi)$ with respect to ξ and b_k are the IC's. Substitution of (1.38) into (1.39) bounces,

$$\Upsilon^n(\xi) = f(\xi) + g(\xi)\int_a^b K(\xi,t)h(t)u(t)dt\,, \qquad \Upsilon^{(k)}(0) = b_{k,} \quad 0 \leq k \leq (n-1) \tag{1.40}$$

We can without much of a stretch see that the definite integral in the IDE (1.39) includes an integrand that be contingent completely on t. So we can say that the definite basic at the R.H.S of (1.39) is identical to a consistent α. As such,

$$\alpha = \int_a^b h(t)u(t)dt \tag{1.41}$$

Subsequently, Eq. (1.39) converts

$$\Upsilon^n(\xi) = f(\xi) + \alpha g(\xi) \tag{1.42}$$

Integration of opposite sides of (1.42) n times from 0 to ξ, and utilizing the endorsed IC's, we can find an articulation for $\Upsilon(\xi)$ that includes the consistent α notwithstanding the variable ξ. This implies we can compose

$$\Upsilon(\xi) = v(\xi; \alpha) \tag{1.43}$$

Substitution of (1.42) into the R.H.S of (1.40), assessing the fundamental, and explaining the subsequent equality, we decide a mathematical incentive for the consistent α. This prompts the solution in exact form of $\Upsilon(\xi)$ got after substitution of the subsequent estimation of α into (1.42). Recall that this technique drives dependably to the precise solution and not to the form of series parts.

1.4 Theorems and Definitions interrelated to fuzzy perceptions

1.4.1 Fuzzy number

Any fuzzy number has an option of representation as a fuzzy subset in R. By defining a function $\Upsilon: R \to [0,1]$ having a characteristic of being bound, convex and normal. Now taking a set E having the collection of all fuzzy numbers which have the property of continuity (upper semi) and compact. The α level set $[\Upsilon]_\rho$ where Υ is representing the collection of fuzzy numbers, $0 < \rho \leq 1$, describe as $[\Upsilon]_\rho = \{t \in R, \Upsilon(t) \geq \rho\}$. The convex property hold for U if $\Upsilon(t) \geq \Upsilon(s) \wedge \Upsilon(r) = \min(\Upsilon(s), \Upsilon(r)), where\ s < t < r$. If $\exists\ t_o \in R$ such that $\Upsilon(t_o) = 1$, then U becomes normal. U is said to be continuous (upper semi) if for every $\varepsilon > 0, \Upsilon^{-1}([0, a + \varepsilon)), \forall\ a \in [0,1]$ is open in the typical topology of R.

Absolute value $|\Upsilon|$ of $\Upsilon \in E$ is describe as

$$|\Upsilon|(t) = max\{\Upsilon(t), \Upsilon(-t)\}, if\ t \geq 0$$
$$= 0, if\ t < 0$$

Now its obvious ρ- level set of U has a property of being close and bound interlude $\left[\underline{Y}(\rho), \overline{Y}(\rho)\right]$ as $\underline{Y}(\rho)$ represents the opposite-hand end point while $\overline{Y}(\rho)$ shows the rightward conclusion point for $[Y]_\rho$ both of random fuzzy numbers $\overline{Y} = \left[\underline{Y}(\rho), \overline{Y}(\rho)\right]$ and $\overline{v} = \left[\underline{V}(\rho), \overline{V}(\rho)\right]$ are same so $\overline{Y} = \overline{V}$ if and only if $\underline{Y}(\rho) = \underline{V}(\rho)$ and $\overline{Y}(\rho) = \overline{V}(\rho)$. every $y \in R$ canister be observed as a fuzzy number $\overline{y}$ represent by

$$\overline{y}(t) = \begin{Bmatrix} 1 & if & t = y \\ 0 & if & t \neq y \end{Bmatrix}$$

$\overline{d}: L(R) \times L(R) \to R$ is a mapping showing the distance between fuzzy numbers and can be shown as

$$d(Y, V) = \sup_{0 \le \rho \le 1} max\left\{\left|\underline{Y}(\rho) - \underline{V}(\rho)\right|, \left|\overline{Y}(\rho) - \overline{V}(\rho)\right|\right\}$$

Where

$$\overline{Y} = \left[\underline{Y}(\rho), \overline{Y}(\rho)\right] \text{ and } \overline{V} = \left[\underline{V}(\rho), \overline{V}(\rho)\right]$$

So,d is a metric on $L(R)$ with the resulting belongings:

1. $d(Y + w, V + w) = d(Y, V)\ for\ all\ Y, V, w \in L(R)$
2. $d(kY, kV) = |k| d(Y, V)$ for all $Y, V, \in L(R)$
3. $d(Y + w, w + e) \le d(Y, w) + d(V, e) for\ all\ Y, V, w, e \in L(R)$
4. $\left(d, L(R)\right)$ is a complete metric space.

1.4.2 Definition:

Consider $f: R \to L(R)$ as a fuzzy valued function f is continuous for $t_o \in R$ for each $\varepsilon > 0$ their exist $\delta > 0$ such that

$$d(\left(f(t), f(t_0)\right) < \varepsilon\ whenever\ |t - t_o| < \delta$$

1.4.3 Definition:

consider $f: R \to L(R)$ as a fuzzy valued function and $\xi_o \in R$ than f is differentiable at ξ_o. If $\exists$ $f'(\xi_o) \in L(R)$ such that

(a) $\lim\limits_{h \to 0+} \frac{f(\xi_o + h) - f(\xi_o)}{h} = \lim\limits_{h \to 0+} \frac{f(\xi_o) - f(\xi_o - h)}{h} = f^{(1)}(\xi_o)$

(b) $\lim_{h\to 0-}\frac{f(\xi_o+h)-f(\xi_o)}{h}=\lim_{h\to 0-}\frac{f(\xi_o)-f(\xi_o-h)}{h}=f^{(1)}(\xi_o)$

Theorem:

consider $f: R\to L(R)$ as a fuzzy valued function and shows $f(t)=\left[\underline{f}(t,\rho),\overline{f}(t,\rho)\right]$for each $0\leq \alpha\leq 1$ these are hold- [28]

(a) If f is differentiable (a) in definition 1.7.2, then $\underline{f}(t,\rho)$ and $\overline{f}(t,\rho)$ are differentiable and $f^{(1)}(t)=\left[\underline{f}^{(1)}(t,\rho),\overline{f}^{(1)}(t,\rho)\right]$

(b) Conditionally the differentiability of f in (b) in definition 1.7.2, then$\underline{f}(t,\rho)$ and $\overline{f}(t,\rho)$ are differentiable and $f^{(1)}(t)=\left[\overline{f}^{(1)}(t,\rho),\underline{f^{(1)}}(t,\rho)\right]$

Theorem:

$f: R\to L(R)$ be the fuzzy valued function and represents $f(t)=\left[\underline{f}(t,\rho),\overline{f}(t,\rho)\right]$for each $0\leq \rho\leq 1$ followings are hold-

(a) If f and $f^{(1)}$is have the property of differentiability in the 1st arrangement of (a) in 1.7.2 or if f and $f^{(1)}$ have property of differentiability in the 2nd arrangement of (b) in 1.7.2, then $\overline{f}^{(1)}(t,\rho)\ and\ \underline{f^{(1)}}(t,\rho)$ are differentiable

$$f^{(2)}(t)=\left[\underline{f^{(2)}}(t,\rho),\overline{f}^{(2)}(t,\rho)\right]$$

(b) If f in (a) and $f^{(1)}$ in (b) has the characteristic of differentiability in the 1st and 2nd arrangement respectively or If f in (b) and $f^{(1)}$ in (a) has the characteristic of differentiability in the 2nd and 1st arrangement respectively in 1.7.2, then the $\overline{f}^{(1)}(t,\rho)\ and\ \underline{f^{(1)}}(t,\rho)$ are differentiable.

Theorem:

A function $f(\xi)$ on $[0,\infty]$ having mapping on R and is represented by $\left[\underline{f}(\xi,\rho),\overline{f}(\xi,\rho)\right]$for any fixed $r\in[0,1]$, suppose $\underline{f}(\xi,\rho),\overline{f}(\xi,\rho)$ are Rimann-integrable on $[a,b]$, for each $b\geq a$ and lease their exist both the positive $\underline{M}(\rho),\overline{M}(\rho)$ such that

$$\int_{a}^{b}\left|\underline{f}(\xi,\rho)\right|d\xi\leq\underline{M}(\rho)\ and\ \int_{a}^{b}\left|\overline{f}(\xi,\rho)\right|d\xi\leq\overline{M}(\rho)$$

for every $b\geq a$. Then $f(\xi)$ is improper fuzzy Rimannintegrable on $[0,\infty]$ and then the $f(\xi)$ is a fuzzy number. Additionally, we can say

$$\int_{a}^{\infty}f(\xi)d\xi=\int_{a}^{b}\underline{f}(\xi,\rho)d\xi\,,\int_{a}^{b}\overline{f}(\xi,\rho)d\xi$$

Proposition:

Consider $f(\xi)$ and $g(\xi)$be a fuzzy on R and fuzzy Riemann-integrable on $I=[a,\infty)$,then $f(\xi)+g(\xi)$ is Rimann-integrable on $I=[a,\infty)$

$$\int_{I}[(f(\xi)+g(\xi)]d\xi=\int_{I}f(\xi)d\xi+\int_{I}g(\xi)d\xi$$

Definition:

The FLT of a fuzzy $f(t)$ on R is defined as follows:

$$f(s)=L\{f(t)\}=\int_{0}^{\infty}e^{-st}f(t)dt=\lim_{T\to\infty}\int_{0}^{T}e^{-st}f(t)dt$$

The sign L is FLT, which turns on fuzzy real valued function $f=f(t)$and produces $f(s)=L\{f(t)$. And the fuzzy Laplace transform for $f(t)$ can be as followed

$$f(s,\rho)=L\{f(t,\rho)\}=[l\{\underline{f}(t,\rho\},l\{\overline{f}\{t,\rho\}]$$

$$\underline{l\{f}(t,\rho\}=\int_{0}^{\infty}e^{-st}\underline{f}(t)dt=\lim_{T\to\infty}\int_{0}^{T}e^{-st}\underline{f}(t)dt$$

$$0\leq\rho\leq1$$

$$l\{\overline{f}\{t,\rho\} = \int_{0}^{\infty} e^{-st}\overline{f}(t)dt = \lim_{T\to\infty} \int_{0}^{T} e^{-st}\overline{f}(t)dt$$

$$0 \le \rho \le 1$$

Definition:

Fuzzy Convolution Theorem:

The convolution for both the $f\ and\ g$ which are fuzzy real value function defined for $t \ge 0$.

$$(f * g)(t) = \int_{0}^{T} f(T)g(t-T)dT$$

Theorem:

Consider $f\ and\ g$ defined on R are continuous (piecewise) on $[0, \infty]$ having exponential order p, then

$$L\{(f * g)(t)\} = L\{f(t)\}L\{g(t)\} = F(s).G(s)$$

1.5 Sumudu transform:

Quite a while prior, differential equations warred a vital part in all parts of connected science and designing fields. In spite of this, they are difficult to discover an answer, either numerically or hypothetically for these equations. So as to grow new systems, help in getting careful and surmised arrangements of these equations is as yet a major issue need new techniques.

Watugula presented another vital change and called it as Sumudu transform. Which is characterized as:

Watugula presented another vital change and called it as Sumudu transform. Which is characterized as:$F(u) = \mathcal{S}[f(t)] = \int_0^\infty \frac{1}{u} e^{\left(-\frac{t}{u}\right)} f(t)dt$

Watugula connected this changes to the arrangement of ordinary differential equations. In light of its valuable properties, the Sumudu transform helps in taking care of complex issues in connected sciences and designing arithmetic. Henceforward, is the meaning of the Sumudu transform and properties portraying the effortlessness of the transform. [12]

1.5.1 Definition of sumudu transform:

The Sumudu transform of the function $f(t)$is defined by:

$$F(u) = \mathcal{S}[f(t)] = \int_0^\infty \frac{1}{u} e^{\left(-\frac{t}{u}\right)} f(t)dt$$

$$F(u) = \mathcal{S}[f(t)] = \int_0^\infty f(ut)e^{-t}dt$$

For any function $f(t)$ and $-\tau_1 < u < \tau_2$

Theorem

If $c_1 \geq 0, c_2 \geq 0$ and $c \geq 0$ are any constant and $f_1(t)$, $f_2(t)$ and $f(t)$ any functions having the Sumudu transform $G_1(u)$, $G_2(u)$ and $G(u)$ respectively then

i. $\mathcal{S}[c_1 f_1(t) + c_2 f_2(t)] = c_1\mathcal{S}[f_1(t)] + c_2\mathcal{S}[f_2(t)] = c_1 G_1(u) + c_2 G_2(u)$
ii. $\mathcal{S}[f(ct)] = G(cu)$
iii. $\lim_{t\to\infty} f(t) = f(0) = \lim_{u\to 0} G(u)$

Further are worded more, for several functions $f(t)$ defined for $t \geq 0$ in the neighbourhood of infinity.

$$\lim_{t\to\infty} f(t) = \lim_{u\to 0} G(u)$$

Theorem

If $\mathcal{S}[f(t)] = F(u)$ and

$$g(t) = \begin{cases} f(t-\tau), & t \geq \tau \\ 0 & t \geq \tau \end{cases}$$

Then,

$$\mathcal{S}[g(t)] = e^{\left(-\frac{t}{u}\right)} G(u)$$

1.5.2 Some important formula for sumudu transform: [24]

	$f(t)$	$G(u)=\mathcal{S}(f(t))$
1	1	1
2	t	u
3	$\frac{t^{n-1}}{(n-1)!}, n=0,1,2,3\ldots$	u^{n-1}
4	$\frac{t^{n-1}}{\Gamma(n)}, n>0$	u^{n-1}
5	e^{at}	$\frac{1}{1-au}$
6	$\frac{(t^{n-1})e^{at}}{(n-1)!}, n=0,1,2,3\ldots$	$\frac{u^{n-1}}{(1-au)^n}$
7	$\frac{sinat}{a}$	$\frac{u}{1+a^2u^2}$
8	$cosat$	$\frac{1}{1+a^2u^2}$
9	$\frac{e^{bt}sinhat}{a}$	$\frac{u}{(1-bu)^2+a^2u^2}$
10	$e^{bt}cosht$	$\frac{1-bu}{((1-bu)^2+a^2u^2)}$
11	$\frac{(e^{bt}-e^{at})}{b-a}, a\neq b$	$\frac{u}{(1-bu)+(1-au)}$

12	$\frac{(be^{bt}-ae^{at})}{b-a}, a \neq b$	$\frac{u}{(1-bu)+(1-au)}$
13	$\frac{tsinat}{2a}$	$\frac{u^2}{(1+a^2u^2)^2}$
14	$cosat - \frac{1}{2}atsinat$	$\frac{1}{(1+a^2u^2)^2}$
15	$tcosat$	$\frac{u((1-a^2u^2)}{(1+a^2u^2)^2}$
16	$\frac{sinat - atcosat}{2a^3}$	$\frac{u^3}{(1+a^2u^2)^2}$
17	$\frac{atcoshat - sinhat}{2a^3}$	$\frac{u^3}{(1+a^2u^2)^2}$
18	$\frac{tsinhat}{2a}$	$\frac{u^2}{(1+a^2u^2)^2}$
19	$\frac{sinhat + tcoshat}{2a}$	$\frac{u}{(1-a^2u^2)^2}$
20	$tcoshat$	$\frac{u((1+a^2u^2)}{(1-a^2u^2)^2}$
21	$\frac{t^2sinat}{2a}$	$\frac{u^3((3-a^2u^2)}{(1+a^2u^2)^3}$
22	$\frac{t^2cosat}{2}$	$\frac{u^2((1-3a^2u^2)}{(1+a^2u^2)^3}$
23	$\frac{t^3cosat}{6}$	$\frac{u^3((1-6a^2u^2+a^4u^4)}{(1+a^2u^2)^4}$

24	$\frac{t^3 sinat}{24a}$	$\frac{u^4((1-a^2u^2)}{(1+a^2u^2)^4}$
25	$\frac{at^2 coshat - tsinhat}{8a^5}$	$\frac{u^4}{(1-a^2u^2)^3}$
26	$\frac{t^2 sinhat}{2a}$	$\frac{u^3((3+a^2u^2)}{(1-a^2u^2)^3}$
27	$cosatcoshat$	$\frac{1}{1+4a^4u^4}$
28	$\frac{1}{2a^2} sinatsinhat$	$\frac{u^2}{1+4a^4u^4}$
29	$\frac{1}{2a}(sinatcoshat + cosatsinhat)$	$\frac{u}{1+4a^4u^4}$
30	$\frac{1}{2a^3}(sinhat - sinat)$	$\frac{u^3}{1-a^4u^4}$
31	$\frac{1}{2a^2}(coshat - cosat)$	$\frac{u^2}{1-a^4u^4}$
32	$\frac{1}{2a}(sinhat - sinat)$	$\frac{u}{1-a^4u^4}$
33	$\frac{1}{2a}(coshat + cosat)$	$\frac{1}{1-a^4u^4}$
34	$\frac{2(cosat - cosbt)}{t}$	$\frac{1}{u} ln \frac{(1+a^2u^2)}{(1+b^2u^2)}$
35	$\frac{(e^{-bt} - e^{-at})}{t}$	$\frac{1}{u} ln \frac{(1+a\,u\,)}{(1+b\,u\,)}$
36	$\frac{t^2 cosat}{2}$	$\frac{u^2((1-3a^2u^2)}{(1+a^2u^2)^3}$

1.5.3 Fuzzy Sumudu transform:

Taking $f: R \to f(R)$ as a continuous fuzzy valued function and undertake that $f(u\xi) \odot e^{-\xi}$ as an improper fuzzy Riemann-integrable on $[0, \infty)$, then [15]

$$\int_0^\infty f(u\xi) \odot e^{-\xi} d\xi$$

is called FST and we can represent it as

$$F(u) = \mathcal{S}[f(\xi)] = \int_0^\infty f(u\xi) \odot e^{-\xi} d\xi, \qquad u \in [\tau_1, \tau_2]$$

$$= \mathcal{S}[f(\xi)] = \left[\mathcal{S}\left[\underline{f}_\alpha(\xi)\right], \mathcal{S}[\overline{f}_\alpha(\xi)]\right]$$

Important theorems and properties:

As $f: R \to f(R)$ is a continuous fuzzy valued function and if $F(u) = \mathcal{S}[f(\xi)]$ then

$$\mathcal{S}\left[f^{(1)}(\xi)\right] = \begin{cases} \dfrac{F(u)}{u} - \dfrac{f(0)}{u} & if\ f\ is\ (i) differentiable\ and\ u > 0 \\ -\dfrac{f(0)}{u} - \dfrac{\left(-F(u)\right)}{u} & if\ f\ is\ (ii) differentiable\ and\ u > 0 \end{cases}$$

Proof: case (i)

let f is differentiable then

$$\frac{F(u)}{u} - \frac{f(0)}{u} = \left[\frac{\mathcal{S}\left[\underline{f}_\rho(\xi)\right]}{u} - \frac{\underline{f}_\rho(0)}{u}, \frac{\mathcal{S}\left[\overline{f}_\rho(x)\right]}{u} - \frac{\overline{f}_\rho(0)}{u}\right]$$

$$= \mathcal{S}\left[\left[\underline{f}_\rho(\xi)\right] \mathcal{S}\left[\overline{f}_\rho(\xi)\right]\right]$$

$$\frac{F(u)}{u} - \frac{f(0)}{u} = \mathcal{S}\left[f^{(1)}(\xi)\right]$$

Proof: case (ii)

let f is differentiable then

$$-\frac{f(0)}{u} - \frac{\left(-F(u)\right)}{u} = \left[-\frac{\underline{f}_\rho(0)}{u} + \frac{\mathcal{S}\left[\underline{f}_\rho(\xi)\right]}{u}, -\frac{\overline{f}_\rho(0)}{u} + \frac{\mathcal{S}\left[\overline{f}_\rho(\xi)\right]}{u}\right]$$

$$= \mathcal{S}\left[\left[\underline{f}_\rho(\xi)\right] \mathcal{S}\left[\overline{f}_\rho(\xi)\right]\right]$$

$$-\frac{f(0)}{u}-\frac{\left(-F(u)\right)}{u}=\mathcal{S}\left[f^{(1)}(\xi)\right]$$

Theorem:

Let $f: R \rightarrow f(R)$ is a continuous fuzzy valued function and if $F(u)=\mathcal{S}[f(x)]$ then

$$\mathcal{S}\left(e^{-a\xi}\odot f(t)\right)=\frac{1}{1+au}F\left(\frac{u}{1+au}\right), \qquad au\neq -1 \ \ and \frac{1}{1+au}>0$$

Proof:

$$\mathcal{S}\left(e^{-a\xi}\odot f(t)\right)=\left[\int_0^\infty \underline{f_\rho}(u\xi)e^{-a(\xi u)}e^{-\xi}d\xi, \int_0^\infty \overline{f_\rho}(u\xi)e^{-a(\xi u)}e^{-\xi}d\xi\right]$$

$$=\left[\int_0^\infty \underline{f_\rho}(u\xi)e^{-a(1+u)\xi}d\xi, \int_0^\infty \overline{f_\rho}(\xi)e^{-a(1+u)\xi}d\xi\right]$$

Now let $v=(1+au)\xi \ \ and \ \ d\xi=\frac{v}{1+au}$

So, the equation develops

$$=\left[\frac{1}{1+au}\int_0^\infty \underline{f_\rho}\left(\frac{uv}{1+au}\right)e^{-v}dv, \frac{1}{1+au}\int_0^\infty \overline{f_\rho}\left(\frac{uv}{1+au}\right)e^{-v}dv,\right]$$

$$=\frac{1}{1+au}\int_0^\infty f_\alpha\left(\frac{uv}{1+au}\right)e^{-v}dv,$$

So, we have verified that

$$\mathcal{S}\left(e^{-a\xi}\odot f(t)\right)=\frac{1}{1+au}F\left(\frac{u}{1-au}\right)$$

Likewise, we can prove

$$\left(e^{a\xi}\odot f(t)\right)=\frac{1}{1-au}F\left(\frac{u}{1-au}\right)$$

Theorem:

consider $f: R \rightarrow f(R)$ is a continuous fuzzy valued function and if $F(u)=\mathcal{S}[f(\xi)]$ then

$$\mathcal{S}\left[\int_0^\xi f(\xi)d\xi\right]=uF(u)$$

Proof:

Assume function h is differentiable, and

$$\underline{h}_\rho(\xi) = \int_0^\xi \underline{f}_\rho(\xi)d\xi\,, \overline{h}_\rho(\xi) = \int_0^\xi \overline{f}_\rho(\xi)d\xi \qquad \underline{h}_\rho(0) = 0 = \overline{h}_\rho(0)\,, \qquad h^{(1)}(\xi) = f(\xi)$$

$$\mathcal{S}\left(h^{(1)}(\xi)\right) = \frac{H(u)}{u} - \frac{h(o)}{u} = \left[\frac{\mathcal{S}\left[\underline{h}_\rho(\xi)\right]}{u} - \frac{\underline{h}_\rho(0)}{u}, \frac{\mathcal{S}\left[\overline{h}_\rho(\xi)\right]}{u} - \frac{\overline{h}_\rho(0)}{u}\right]$$

$$= \left[\frac{\mathcal{S}\left[\underline{h}_\rho(\xi)\right]}{u}, \frac{\mathcal{S}\left[\overline{h}_\rho(\xi)\right]}{u}\right]$$

$$= \left[\frac{1}{u}\,\mathcal{S}\int_0^\xi \underline{f}_\rho(\xi)d\xi\,, \frac{1}{u}\mathcal{S}\int_0^\xi \overline{f}_\rho(\xi)d\xi\right]$$

So, proved

$$\mathcal{S}\left[\int_0^\xi f(\xi)d\xi\right] = uF(u)$$

Chapter No 2

Solution of Fuzzy Integro-Differential Equation with the use of Fuzzy Laplace Transformation

The concept of fuzzy sets and set operations was first introduced by Zadeh and subsequently several authors have studied various aspects of the theory and applications of fuzzy sets. Abbasbandy et. al introduced a numerical algorithm for solving linear Fredholm fuzzy integral equations of second kind by using parametric form of fuzzy number and converting a linear fuzzy Fredholm integral equations to two linear systems of integral equation of the second kind in crisp case Badolian et. al studied another numerical method for solving linear fuzzy Fredholm integral equation of second kind by using Adomian method. Moreover, Friedman et. al investigated an emending method to solve fuzzy Voltera and Fredholm integral equations. However, many other authors obtained the numerical integration of fuzzy valued functions and solving fuzzy Voltera and Fredholm equations. The concept of fuzzy Laplace Transformation was introduced by Allahviranloo, Ahmadi After many other researchers use it to solve fuzzy differential equations, fuzzy integral equations etc. Some authors discussed the solution of fuzzy integro-differential equation by fuzzy differential transform method in their research paper. [16,19,20]

In this section we examine the strategy for explaining fuzzy integro-differential equations under certain condition by utilizing fuzzy Laplace transform. And will see the different results and their behavior at long last we will give some illustrative numerical examples. [17,18]

2.1 Main Construction:

The general Volterra integro-differential equation is [7,13]

$$\Upsilon^{n}(\xi) = f(\xi) - \int_{0}^{\xi} k(\xi - t)\Upsilon(t)dt \tag{2.1}$$

with

$$\Upsilon^{k}(0) = \overline{a} = \left(\underline{a^{k}}, \overline{a^{k}}\right); 0 \leq k \leq n-1$$

the nth derivative of Υ is denoted by Υ^{n}.

By applying fuzzy Laplace transform on both sides of (2.1),

$$L[\Upsilon^{n}(\xi)] = L[f(\xi)] - L[\int_{0}^{\xi} k(\xi - t)]\Upsilon(t)dt \tag{2.2}$$

Use of FLT and some of its properties will give u

$$s^{n}L\big(\Upsilon(\xi)\big) - s^{n-1}\Upsilon(0) - s^{n-2}\Upsilon^{(1)}(0) \dots\dots\dots - \Upsilon^{(n-1)}(0) = L[f(\xi)] - L[\int_{0}^{\xi} k(\xi - t)]\Upsilon(t)dt$$

$$s^{n}L\left(\underline{\Upsilon}(\xi,\rho)\right) - s^{n-1}\underline{a_{o}} - s^{n-2}\underline{a_{1}} \dots\dots\dots - \underline{a_{n-1}}(0) = L\{\underline{f}(\xi,\rho\} + L[\underline{k}(\xi;\rho)]L[\underline{\Upsilon}(\xi;\rho)]$$

$$0 \le \rho \le 1 \tag{2.3}$$

Where,

$$\Upsilon(0) = \underline{a_{o}}, \Upsilon'(0) = \underline{a_{1}}, \dots \Upsilon^{n-1}(0) = \underline{a_{n-1}} \tag{2.4}$$

$$s^{n}L\left(\overline{\Upsilon}(\xi,\rho)\right) - s^{n-1}\overline{a_{o}} - s^{n-2}\overline{a_{1}} \dots\dots\dots - \overline{a_{n-1}}(0) = L\{\overline{f}(\xi,\rho\} + L[\overline{k}(\xi;\rho)]L[\overline{\Upsilon}(\xi;\rho)]$$

Where,

$$\Upsilon(0) = \overline{a_{o}}, \Upsilon'(0) = \overline{a_{1}}, \dots \Upsilon^{n-1}(0) = \overline{a_{n-1}}(0)$$

$$0 \le \rho \le 1 \tag{2.5}$$

Now we discuss the following cases

(i) if $\Upsilon(\xi;\rho)$ and $k(\xi;\rho)$ both are positive

$$L\big[\underline{k}(\xi;\rho)\big]L\big[\underline{\Upsilon}(\xi;\rho)\big] = L\big[\underline{k}(\xi;\rho)\big]L\big[\underline{\Upsilon}(\xi;\rho)\big]$$

$$L\big[\overline{k}(\xi;\rho)\big]L\big[\overline{\Upsilon}(\xi;\rho)\big] = L\big[\overline{k}(\xi;\rho)\big]L\big[\overline{\Upsilon}(\xi;\rho)\big]$$

(ii) if $\Upsilon(\xi;\rho)$ is negative and $k(\xi;\rho)$ is positive

$$L\big[\underline{k}(\xi;\rho)\big]L\big[\underline{\Upsilon}(\xi;\rho)\big] = L\big[\overline{k}(\xi;\rho)\big]L\big[\underline{\Upsilon}(\xi;\rho)\big]$$

$$L\big[\overline{k}(\xi;\rho)\big]L\big[\overline{\Upsilon}(\xi;\rho)\big] = L\big[\underline{k}(\xi;\rho)\big]L\big[\overline{\Upsilon}(\xi;\rho)\big]$$

(iii) if $\Upsilon(\xi;\rho)$ is positive and $k(\xi;\rho)$ is negative

$$L\big[\underline{k}(\xi;\rho)\big]L\big[\underline{\Upsilon}(\xi;\rho)\big] = L\big[\underline{k}(\xi;\rho)\big]L\big[\overline{\Upsilon}(\xi;\rho)\big]$$

$$L\big[\overline{k}(\xi;\rho)\big]L\big[\overline{\Upsilon}(\xi;\rho)\big] = L\big[\overline{k}(\xi;\rho)\big]L\big[\underline{\Upsilon}(\xi;\rho)\big]$$

(iv) if $\Upsilon(\xi;\rho)$ and $k(\xi;\rho)$ both are negative

$$L\big[\underline{k}(\xi;\rho)\big]L\big[\underline{\Upsilon}(\xi;\rho)\big] = L\big[\overline{k}(\xi;\rho)\big]L\big[\overline{\Upsilon}(\xi;\rho)\big]$$

$$L\big[\overline{k}(\xi;\rho)\big]L\big[\overline{\Upsilon}(\xi;\rho)\big] = L\big[\underline{k}(\xi;\rho)\big]L\big[\underline{\Upsilon}(\xi;\rho)\big]$$

We will explore the (i). All the other cases are same.[25]

$$s^n L[\underline{Y}(\xi;\rho)] - s^{n-1}\underline{a_0} - s^{n-2}\underline{a_1} \dots \dots . \underline{a_{n-1}} = L\left\{\underline{f}(\xi;\rho)\right\} + L[\underline{k}(\xi;\rho)]L[\underline{Y}(\xi;\rho)]$$

$$0 \le \rho \le 1$$

(2.6)

and,

$$s^n L[\underline{Y}(\xi;\rho)] - s^{n-1}\overline{a_o} - s^{n-2}\overline{a_1} \dots \dots . -\overline{a_{n-1}} = L[\overline{f}(\xi;\rho)] + L[\overline{k}(\xi;\rho)]L[\overline{Y}(\xi;\rho)]$$

(2.7)

by simplification

$$s^n L[\underline{Y}(\xi;\rho)] - L[\underline{k}(\xi;\rho)]L[\underline{Y}(\xi;\rho)] = L\left[\underline{f}(\xi;\rho)\right] + s^{n-1}\underline{a_0} + s^{n-2}\underline{a_1} \dots \dots . \underline{+a_{n-1}}$$

$$s^n L[\underline{Y}(\xi;\rho)]\left(1 - L[\underline{k}(\xi;\rho)]\right) = L\left[\underline{f}(\xi;\rho)\right] + s^{n-1}\underline{a_0} + s^{n-2}\underline{a_1} \dots \dots . \underline{+a_{n-1}}$$

(2.8)

Similarly,

$$s^n L[\overline{Y}(\xi;\rho)] - L[\overline{k}(\xi;\rho)]L[\overline{Y}(\xi;\rho)] = L[\overline{f}(\xi;\rho)] + s^{n-1}\overline{a_o} + s^{n-2}\overline{a_1} \dots \dots . +\overline{a_{n-1}}$$

$$s^n L[\overline{Y}(\xi;\rho)](1 - L[\underline{k}(\xi;\rho)] = L[\overline{f}(\xi;\rho)] + s^{n-1}\overline{a_o} + s^{n-2}\overline{a_1} \dots \dots . +\overline{a_{n-1}}$$

In compact form:

$$L[\underline{Y}(\xi;\rho)] = \frac{L\left[\underline{f}(\xi;\rho)\right] + s^{n-1}\underline{a_0} + s^{n-2}\underline{a_1} \dots \dots . \underline{\underline{a_{n-1}}}}{\left(s^n - L[\underline{k}(\xi;\rho)]\right)}$$

$$0 \le \rho \le 1$$

And,

$$L[\overline{Y}(\xi;\rho)] \quad = \frac{L[\overline{f}(\xi;\rho)] + s^{n-1}\overline{a_o} + s^{n-2}\overline{a_1} \dots \dots . +\overline{a_{n-1}}}{\left(s^n - L[\overline{k}(\xi;\rho)]\right)}$$

$$0 \le \rho \le 1$$

Now applying the L^{-1} on the both sides of the equation above equation. we can without much of a stretch get the estimation of $\underline{Y}(\xi;\rho)$ and $\overline{Y}(\xi,\rho)$ where $0 \le \rho \le 1$

Numerical problems

Example 2.1: The FVID equation is

$$\Upsilon^{(2)}(\xi) = (\rho + 2{,}4 - \rho)\xi + \int_{0}^{\xi} (\xi - t)\Upsilon(t)dt$$

$$\Upsilon(0) = (\rho + 1{,}3 - \rho); \Upsilon^{(1)}(0) = (\alpha, 2 - \alpha) \tag{2.9}$$

The use of fuzzy Laplace transforms on (2.9) will give us

$$L\left[\Upsilon^{(2)}(\xi)\right] = L[(\rho + 2{,}4 - \rho)\xi] + L[\xi]L[\Upsilon(\xi)]$$

$$s^2 L\{\Upsilon(\xi)\} - s\big(\Upsilon(0)\big) - \Upsilon^{(1)}(0) = L\big\{\big((\rho + 2{,}4 - \rho)\xi\big)\big\} + L\{\xi\}L\{(\Upsilon(\xi)\} \tag{2.10}$$

$$s^2 L\{\underline{\Upsilon}(\xi;\rho)\} = L\{\ (\rho + 2)\xi\} + L\{\xi\}L\{\underline{\Upsilon}(\xi;\rho)\} + s(\rho + 1) + \rho$$

$$\Upsilon(0) = \rho + 1, \Upsilon^{(1)}(0) = \rho \tag{2.11}$$

$$s^2 L\{\overline{\Upsilon}(\xi;\rho)\} = L\{\ (4 - \rho)\xi\} + L\{\xi\}L\{\overline{\Upsilon}(\xi;\rho)\} + s(3 - \rho) + (2 - \rho)$$

$$\Upsilon(0) = 3 - \rho, \Upsilon^{(1)}(0) = 2 - \rho \tag{2.12}$$

By simplification of (2.11) and (2.12)

$$L\{\underline{\Upsilon}(\xi;\rho)\} = (\rho + 2)\left(\frac{1}{s^4 - 1}\right) + (\rho + 1)\left(\frac{s^3}{s^4 - 1}\right) + \rho\left(\frac{s^2}{s^4 - 1}\right) \tag{2.13}$$

$$L\{\overline{\Upsilon}(\xi;\rho)\} = (4 - \rho)\left(\frac{1}{s^4 - 1}\right) + (3 - \rho)\left(\frac{s^3}{s^4 - 1}\right) + (2 - \rho)\left(\frac{s^2}{s^4 - 1}\right) \tag{2.14}$$

Now by applying the inverse fuzzy Laplace transform one two sides of the (2.13) and (2.14)

$$L^{-1}\left[L\{\underline{\Upsilon}(\xi;\rho)\}\right] = L^{-1}\left[(\rho + 2)\left(\frac{1}{s^4 - 1}\right) + (\rho + 1)\left(\frac{s^3}{s^4 - 1}\right) + \rho\left(\frac{s^2}{s^4 - 1}\right)\right]$$

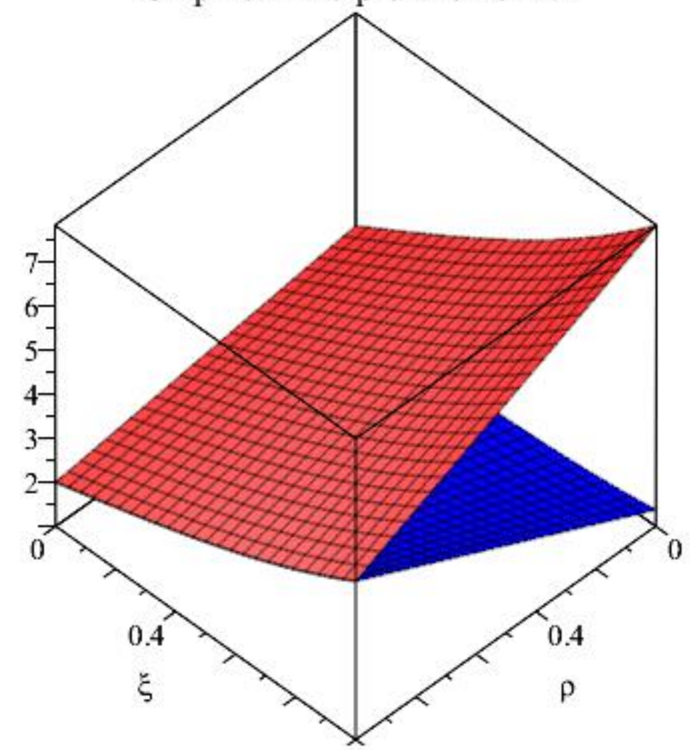
$$L^{-1}\left[L\{\overline{\Upsilon}(\xi;\rho)\}\right] = L^{-1}\left[(4 - \rho)\left(\frac{1}{s^4 - 1}\right) + (3 - \rho)\left(\frac{s^3}{s^4 - 1}\right) + (2 - \rho)\left(\frac{s^2}{s^4 - 1}\right)\right]$$

Gives,

$$\underline{Y}(\xi;\rho) = (\rho+2).\frac{1}{2}(sinh\xi - sin\xi) + (\rho+1).\frac{1}{2}(cos\xi + cosh\xi) + (\rho)(sin\xi + sinh\xi)$$

$$\overline{Y}(t;\rho) = (4-\rho).\frac{1}{2}(sinh\xi - sin\xi) + (3-\rho).\frac{1}{2}(cos\xi + cosh\xi) + (2-\rho)(sin\xi + sinh\xi) \qquad 0 \le \rho \le 1$$

Graphical interpretation of 2.1



Chapter No 3

Use of sumudu decomposition method for solution of different Volterra integro-differential equation

This technique is to acquire approximate solutions for nonlinear scheme of VID equations through the assistance of SDM. The procedure depends on the use of Sumudu transform to nonlinear coupled VID equation. Nonlinear part of the equation can be solved with the use of Adomian polynomials. We represent which were gotten with the assistance of Adomian decomposition method method(ADM).

The linear and nonlinear Volterra integro-differential equations emerge in numerous logical fields, for example, the populace dynamics, spread of pandemics and semiconductor gadgets. The researchers in various parts of science have been attempting to take care of this sort of issues; be that as it may, finding a correct arrangement isn't simple because of the nonlinear piece of these sort gatherings of conditions. Distinctive analytical techniques have been produced and connected to find the solutions. For instance, Adomian has presented a supposed decomposition method for the solution of arithmetical, differential, integro-differential, differential-deferral and partial differential equation. In the nonlinear case for ordinary differential equation and equations which depend more than one variable, the technique takes the benefit of managing specifically by the problems. Such conditions are illuminated deprived of changing them to equivalent form which is increasingly simple. The method minimizes linearization, perturbation, discretization, or any nonreal supposition. It was additionally proposed in that is the repeated terms show up only for inhomogeneous equation. Therefore, furthermost as of late Wazwaz built up a vital condition that is Fundamentally expected to guarantee the presence of "noise terms" in the inhomogeneous circumstances. The vital change has been utilized to explain a wide range of sorts of differential and integro-differential equations. For comparative issues, Sumudu transform was acquainted and further connected with a few ODEs just as PDEs. [9,13,14]

There are some important and interesting properties we will discuss below.

$$f(t)=\sum_{n=0}^{\infty}a_n t^n \quad then \quad F(u)=\sum_{n=0}^{\infty}n!\,a_n u^n \tag{3.1}$$

the close association between STD technique emerging for getting the solution of nonlinear VID condition is illustrated. [22,23]

Amid the investigation, we use the same transform which is characterized over the arrangement of the accompanying functions:

$$A = \left\{ f(t) : \exists M, \tau_{1,}\tau_{2,} > 0, |f(t)| < Me^{\frac{t}{\tau_j}}, if\ t\epsilon(-1)^j \times [0,\infty) \right\} \tag{3.2}$$

Use of given property

$$G(u) = s[f(t);u]$$
$$= \int_0^\infty f(ut)e^{-t}dt, u \in (\tau_{1,}\tau_2)$$

3.1 Main Method:

We consider general nonlinear Volterra integro-differential equation:

$$\frac{d^n \Upsilon}{d\xi^n} = f(\xi) + \int_0^\xi k(\xi - t)F\big(\Upsilon(t)\big)dt \tag{3.3}$$

For getting the solution of nonlinear VIE by the use of Sumudu transform technique, it is needed to use Sumudu transforms of $\frac{d^n \Upsilon}{d\xi^n}$. So, the results will be given below as

$$\mathcal{S}\left[\frac{d^n u}{d\xi^n}\right] = \frac{1}{u^n}\mathcal{S}[\Upsilon(\xi)] - \frac{1}{u^n}\Upsilon(0) - \frac{1}{u^{n-1}}\Upsilon^{(1)}(0) - \cdots - \frac{\Upsilon^{(n-1)}(0)}{u} \tag{3.4}$$

Applying Sumudu transform to both sides of (3.3) gives

$$\frac{1}{u^n}\mathcal{S}[\Upsilon(\xi)] - \frac{1}{u^n}\Upsilon(0) - \frac{1}{u^{n-1}}\Upsilon^{(1)}(0) - \cdots - \frac{\Upsilon^{(n-1)}(0)}{u}$$
$$= \mathcal{S}[f(\xi)] + u\mathcal{S}(\,k(\xi - t))\mathcal{S}(F(\Upsilon(t)) \tag{3.5}$$

By arrangements we have,

$$\mathcal{S}[\Upsilon(\xi)] \;=\; u^n\mathcal{S}[f(\xi)] + \;\Upsilon(0) + u\Upsilon'(0) + \cdots + u^{n-1}\Upsilon^{n-1}(0)$$
$$+u^{n+1}\,\mathcal{S}(\,k(\xi - t))\,\mathcal{S}(F\big(\Upsilon(t)\big) \tag{3.6}$$

The next step of is to compose the results in the form of infinite technique that is

$$\Upsilon(\xi,\lambda) = \Upsilon_0 + \Upsilon_1 + \Upsilon_2 + \cdots \ldots \Upsilon_n \ldots \ldots \ldots \Upsilon_\infty$$

$$\Upsilon(\xi,\lambda) = \sum_{i=0}^{\infty} \Upsilon_i(\xi) \tag{3.7}$$

And the nonlinear part will be handled in such a way

$$F(\Upsilon(t)) = \sum_{i=0}^{\infty} A_i \tag{3.8}$$

Where A_i are Adomian polynomials of $\Upsilon_O, \Upsilon_1, \Upsilon_2 \ldots, \Upsilon_n$, And they can be calculated by the following formula:

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}[F(\sum_{i=0}^{\infty} \lambda^i \Upsilon_i)]_{\lambda=0} \quad , \qquad i = 0,1,2, \ldots. \tag{3.9}$$

And the solution for all the Adomian polynomials A_n for every arrangement of nonlinearity can be estimated by General formula (3.9) can be easily utilized. supposing that the nonlinear function is $F(\mathrm{U}(\xi))$, that’s why, by using (3.9), Adomian polynomials can be derived as

$$A_o = F(\Upsilon_o), \quad A_1 = \Upsilon_1 F'(\Upsilon_o)$$

$$A_2 = \Upsilon_2 F'(\Upsilon_o) + \frac{1}{2!}\Upsilon_1^2 F''(\Upsilon_o)$$

$$A_3 = \Upsilon_3 F'(\Upsilon_o) + \Upsilon_1 \Upsilon_2 F''(\Upsilon_o) + \tfrac{1}{3!}\Upsilon_1^3 F'''(\Upsilon_o)$$

$$A_4 = \Upsilon_4 F'(\Upsilon_o) + \left(\frac{1}{2!}\Upsilon_2^2 + \Upsilon_1\Upsilon_3\right) F''(\Upsilon_o) + \frac{1}{2!}\Upsilon_1^2\, \Upsilon_2 F'''(\Upsilon_o) + \frac{1}{4!}\Upsilon_1^4\, \Upsilon_2 F^{(iv)}(\Upsilon_o) \tag{3.10}$$

Substitution of (3.7) and (3.8) into (3.6) yields

$$\mathcal{S}[\sum_{i=0}^{\infty} \Upsilon_i(\xi)] = u^n \mathcal{S}[f(\xi)] + \Upsilon(0) + u\Upsilon'(0) + \cdots + u^{n-1}\Upsilon^{n-1}(0)$$

$$+u^{n+1}\, \mathcal{S}\big(\, k(\xi - t)\big)\mathcal{S}([\sum_{i=0}^{\infty} A_i)] \tag{3.11}$$

By comparing the both sides of (3.11) and utilizing standard ADM we will have

$$\mathcal{S}[\Upsilon_o(\xi)] = u^n \mathcal{S}[f(\xi)] + \Upsilon(0) + u\Upsilon'(0) + \cdots + u^{n-1}\Upsilon^{n-1}(0) \tag{3.12}$$

$$\mathcal{S}[\Upsilon_1(\xi)] = u^{n+1}\, \mathcal{S}\big(\, k(\xi - t)\big)\, \mathcal{S}\big(A_O(\xi)\big),$$

$$\mathcal{S}[\Upsilon_2(\xi,t)] = u^{n+1}\,\mathcal{S}\big(\,k(\xi-t)\big)\,\mathcal{S}\big(A_1(\xi)\big)..$$

More generally way, we have

$$\mathcal{S}[\Upsilon_{i+1}(\xi,t)] = u^{n+1}\,\mathcal{S}\big(\,k(\xi-t)\big)\,\mathcal{S}\big(A_i(\xi)\big)\quad i \geq 0.$$

Combined STAD method for getting the solution of nonlinear VIDE of the 2nd type will be exemplified by reviewing the succeeding case.

Numerical problems:

Example 3.1

The nonlinear VIDE is given below.

$$\Upsilon^{(2)}(x) = -1 - \frac{1}{3}(sin\xi + \sin(2\xi)) + cos\xi + \int_0^\xi \sin(\xi-t)\,\Upsilon^2\,(t)dt$$

$$\Upsilon'(0) = -1, \Upsilon(0) = 1 \tag{3.13}$$

By applying Sumudu transform to both sides of (3.13) we obtain

$$\frac{1}{u^2}\mathcal{S}[\Upsilon(x)] - \frac{1}{u^2}\Upsilon(0) - \frac{1}{u}\Upsilon^{(1)}(0) = -1 - \frac{1}{3}\left(\frac{u}{1+u^2} + \frac{2u}{1+4u^2} + \frac{1}{1+u^2}\right) + \frac{u^2}{1+u^2}\mathcal{S}[\Upsilon^2(t)] \tag{3.14}$$

Or equivalently,

$$\mathcal{S}[\Upsilon(\xi)] = 1 - u - u^2 - \frac{1}{3}\left(\frac{u^3}{1+u^2} + \frac{2u^3}{1+4u^2} + \frac{u^2}{1+u^2}\right) + \frac{u^4}{1+u^2}\mathcal{S}[\Upsilon^2(t)] \tag{3.15}$$

Putting the values for $\Upsilon(s)$ and Adomian polynomials for $\left(\Upsilon^2(t)\right)$ in (3.14) and (3.15), respectively, by utilizing the recursive relation equation (3.15), we will get

$$\mathcal{S}[\Upsilon(\xi)] = 1 - u - u^2 - \frac{1}{3}\left(\frac{u^3}{1+u^2} + \frac{2u^3}{1+4u^2} + \frac{u^2}{1+u^2}\right) \tag{3.16}$$

Recall that Adomian polynomials for $F(u(x)) = \Upsilon^2(x)$ are given by

$$A_O = \Upsilon_0^2, \qquad A_1 = 2\Upsilon_0\Upsilon_{1,}$$

$$A_2 = 2\Upsilon_0\Upsilon_2 + \Upsilon_1^2, \quad A_3 = 2\Upsilon_0\Upsilon_3 + 2\Upsilon_1 \tag{3.17}$$

Taking inverse sumudu transform on (3.16) and it will give us

$$\Upsilon_0(\xi) = -1 + \xi + \frac{1}{2}\xi^2 - \frac{1}{6}\xi^3 - \frac{1}{12}\xi^4 + \frac{1}{40}\xi^5 + \frac{1}{360}\xi^6 - \frac{11}{5040}\xi^7 + \cdots$$

$$\Upsilon_1(\xi) = \frac{1}{24}\xi^4 - \frac{1}{60}\xi^5 - \frac{1}{720}\xi^6 + \frac{1}{504}\xi^7 \ldots \tag{3.18}$$

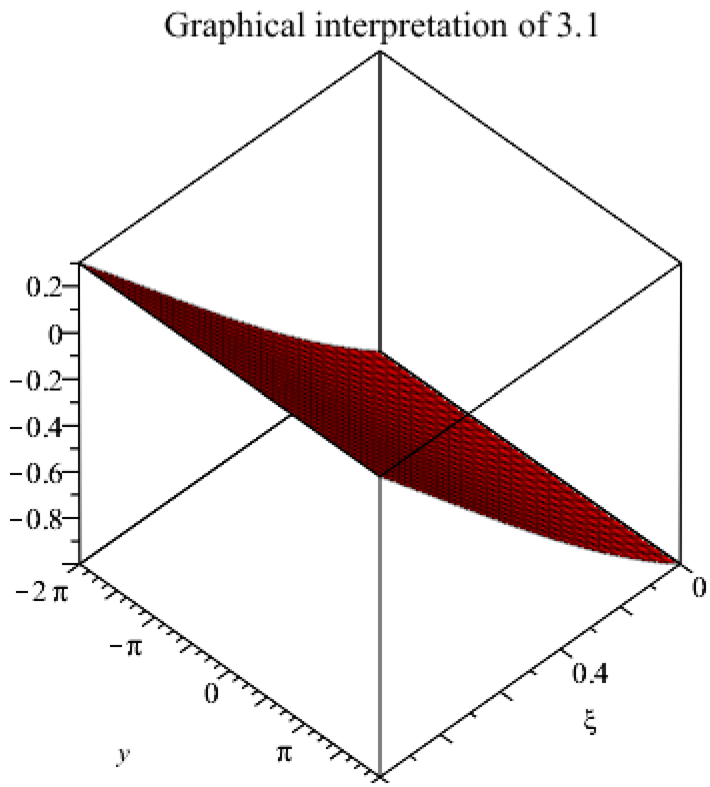
we obtain the series solution as follows

$$\Upsilon(\xi) = \left(\xi - \frac{1}{3!}\xi^3 + \frac{1}{5!}\xi^5 - \frac{1}{7!}\xi^7 + \cdots\right) - \left(1 - \frac{1}{4!}\xi^4 + \frac{1}{6!}\xi^6 - \frac{1}{8!}\xi^8 + \cdots\right) \tag{3.19}$$

The exact result is specified as

$$\Upsilon(\xi) = \sin(\xi) - \cos(\xi) \tag{3.20}$$

In the next problem, we apply the combined STAD method.



3.2 The nonlinear VIDE of the 1st kind

$$\int_0^{\xi} k_1(\xi - t)F\big(\Upsilon(t)\big)dt + \int_0^{\xi} k_2(\xi - t)\Upsilon^n(t)dt = f(\xi) \tag{3.21}$$

Applying Sumudu transform on (3.21) we will obtain

$$\mathcal{S}\left(k_1(\xi) * F\big((\Upsilon)\big)\right) + \mathcal{S}(k_2(\xi) * \Upsilon^n(\xi)) = \mathcal{S}(f(\xi)) \tag{3.22}$$

$$uk_1(u)\mathcal{S}F\big((\Upsilon(\xi)))\big) + uk_2(u)\mathcal{S}\big((\Upsilon^n(\xi)\big) = F(u)\mathcal{S}[\Upsilon(x)] \tag{3.23}$$

$$= u^{n-1}\left(\frac{F(u) + k_2(u)\psi(u) - uk_1(u)\mathcal{S}\left(F\left(\Upsilon(\xi)\right)\right)}{k_2(u)}\right) \tag{3.24}$$

Where

$$\psi(u) = \frac{1}{u^{n-1}}\Upsilon(0) + \frac{1}{u^{n-2}}\Upsilon'(0) + \cdots + \Upsilon^{n-1}(0) \tag{3.25}$$

We can apply Adomian decomposition method to handle (3.24) Substituting (3.10) and (3.11) into (3.24),

$$\mathcal{S}[\sum_{i=0}^{\infty}\Upsilon_i(\xi)] = \frac{u^{n-1}\mathcal{S}[f(\xi)]}{k_2(u)} + \Upsilon(0) + u\Upsilon'(0) + \cdots + u^{n-1}\Upsilon^{n-1}(0) - u^n\frac{k_1(u)}{k_2(u)}\mathcal{S}\left(\left[\sum_{i=0}^{\infty}A_i\right]\right) \tag{3.26}$$

The use of the following recursive relation will give us

$$\Upsilon_0(\xi) = \frac{u^{n-1}\mathcal{S}[f(\xi)]}{k_2(u)} + \Upsilon(0) + u\Upsilon'(0) + \cdots + u^{n-1}\Upsilon^{n-1}(0) \tag{3.27}$$

$$\Upsilon_{k+1}(\xi) = -u^n\frac{k_1(u)}{k_2(u)}\mathcal{S}(A_{k)} \qquad k \geq 0 \tag{3.28}$$

Example.3.3

The nonlinear VIDE of the 1st kind is assumed below solving by the combined STAD method

$$\int_0^{\xi}(\xi - t)\Upsilon^2(t)dt + \int_0^{\xi} e^{(\xi-t)}\Upsilon^{(1)}(t)dt = -\frac{1}{4} - \frac{1}{2}\xi + xe^{\xi} + \frac{1}{4}e^{2\xi}\ , \Upsilon(0) = 1 \tag{3.29}$$

By applying Sumudu transforms to both sides of (3.29), we have

$$\mathcal{S}\left(\Upsilon(\xi)\right) = 1 - \frac{1-u}{4} - \frac{1}{2}u(1-u) + \frac{u}{(1-u)} + 1 - \frac{u}{4(1-2u)} - u^2(1-u)\mathcal{S}[\Upsilon^2(\xi)] \tag{3.30}$$

The series assumption for $\Upsilon(\xi)$ and Adomian polynomials for $\Upsilon^2(\xi)$ in (3.30) gives

$$\mathcal{S}\left(\Upsilon_0(\xi)\right) = 1 - \frac{1-u}{4} - \frac{1}{2}u(1-u) + \frac{u}{(1-u)} + 1 - \frac{u}{4(1-2u)} \tag{3.31}$$

$$\mathcal{S}\left(\Upsilon_{k+1}(\xi)\right) = -u^2(1-u)\mathcal{S}[\Upsilon^2(\xi)] \tag{3.32}$$

By inverse Sumudu transform for the both of the sides of (3.31) and with the help of recursive relation equation (3.32) give

$$\Upsilon_0(\xi) = 1 + \xi + \xi^2 + \frac{1}{3}\xi^3 + \frac{1}{8}\xi^4 + \frac{1}{24}\xi^5 + \cdots$$

$$\Upsilon_1(\xi) = -\frac{1}{2}\xi^2 - \frac{1}{6}\xi^3 - \frac{1}{6}\xi^4 - \frac{1}{12}\xi^5 - \cdots,$$

$$\Upsilon_2(x) = \frac{1}{12}\xi^4 + \frac{1}{20}\xi^5 + \qquad (3.33)$$

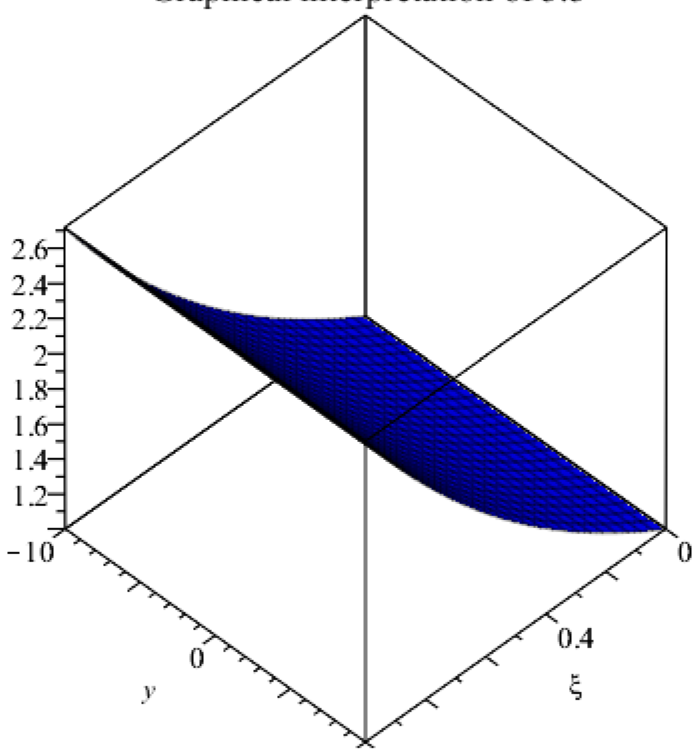
The series solution is given by

$$\Upsilon(\xi) = 1 + \xi + \frac{1}{2!}\xi^2 + \frac{1}{3!}\xi^3 + \frac{1}{4!}\xi^4 + \cdots \qquad (3.34)$$

And the exact solution is

$$\Upsilon(x) = e^{\xi}$$

Graphical interpretation of 3.3



3.4 System of nonlinear VIDE.

In the study of this part, we will examine frameworks of nonlinear VIDE for the 2nd kind by combined STAD method. Consider frameworks of nonlinear VIDE for the 2nd type as pursues:

$$\Upsilon^n(\xi) = f_1(\xi) + \int_0^{\xi} \left(K_1(\xi - t)F_1\big(\Upsilon(t)\big) + R_1(\xi - t)G_1\big(V(t)\big)_1\right) dt,$$

$$V^n(\xi) = f_2(\xi) + \int_0^{\xi} \Big(K_2(\xi - t)F_2\big(\Upsilon(t)\big) + R_2(\xi - t)G_2\big(V(t)\big)\Big)\, dt. \tag{3.35}$$

Applying Sumudu transforms to both sides of above equation, we have

$$\frac{1}{u^n}\mathcal{S}[\Upsilon(\xi)] - \frac{1}{u^n}\Upsilon(0) - \frac{1}{u^{n-1}}\Upsilon'(0) - \cdots - \frac{\Upsilon^{n-1}(0)}{u}$$
$$= \mathcal{S}[f_1(\xi)] + \mathcal{S}(\,k_1(\xi) * F_1\big(\Upsilon(t)\big) + R_1(\xi) * G_1\big(V(\xi)\big)$$
$$\frac{1}{u^n}\mathcal{S}[V(\xi)] - \frac{1}{u^n}V(0) - \frac{1}{u^{n-1}}V'(0) - \cdots - \frac{V^{n-1}(0)}{u}$$
$$= \mathcal{S}[f_2(\xi)] + \mathcal{S}(\,K_2(\xi) * F_2\big(\Upsilon(t)\big) + R_2(\xi) * G_2\big(V(\xi)\big) \tag{3.36}$$

After rearrangement, we get

$$\mathcal{S}[\Upsilon(\xi)] \;= \Upsilon(0) + u\Upsilon'(0) + \cdots + u^{n-1}\Upsilon^{n-1}(0) + u^n\mathcal{S}[f_1(\xi)] + \;\mathcal{S}(\,k_1(\xi) * F_1\big(\Upsilon(t)\big)$$
$$+ R_1(\xi) * G_1\big(V(x)\big)$$
$$\mathcal{S}[V(\xi)] \;= V(0) + uV'(0) + \cdots + u^{n-1}V^{n-1}(0) + u^n\mathcal{S}[f_2(\xi)] + \;\mathcal{S}(\,K_2(\xi) * F_2\big(u(t)\big)$$
$$+ R_2(\xi) * G_2\big(V(\xi)\big) \tag{3.37}$$

To defeat the trouble of the relations involving the nonlinear $F_i\big(\Upsilon(\xi)\big), i = 1{,}2$,we use ADM for solving (3.35) and (3.36). For getting our solution, firstly we introduce linear terms $\Upsilon((\xi)$ and V (ξ) in left side with the help of infinite series of components.

$$\Upsilon(\xi) = \sum_{i=0}^{\infty} \Upsilon_i\,(\xi), \qquad V(\xi) = \sum_{i=0}^{\infty} V_i\,(\xi) \tag{3.38}$$

Also, the nonlinear part $F_i\big(u(\xi)\big)$ in the right side of (3.35) and (3.36) by

$$F_i\big(\Upsilon(t)\big) = \sum_{n=0}^{\infty} A_n$$
$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[F(\sum_{i=0}^{n} \lambda^i \Upsilon_i)]_{\lambda=0} \tag{3.39}$$

Where Adomian polynomials An, $n \geq 0$, can be obtained for all forms of nonlinearity. Substituting (3.38) and (3.39) into (3.35) and (3.36) leads to

$$\mathcal{S}\left[\sum_{n=0}^{\infty} \Upsilon_i\,(\xi)\right] = \Upsilon(0) + u\Upsilon'(0) + \cdots + u^{n-1}\Upsilon^{n-1}(0) + u^n\mathcal{S}[f_1(\xi)]$$

$$+\ u^n\mathcal{S}\big(\,K_1(\xi)\big)\mathcal{S}\left(\left[\sum_{n=0}^{\infty} A_n\right]\right) + u^n\mathcal{S}\big(\,R_1(\xi)\big)\mathcal{S}\left(\left[\sum_{n=0}^{\infty} \tilde{A}_n\right]\right)$$

(3.40)

$$\mathcal{S}\left[\sum_{n=0}^{\infty} V_i\,(\xi)\right] = V(0) + uV'(0) + \cdots + u^{n-1}V^{n-1}(0) + u^n\mathcal{S}[f_2(\xi)]$$

$$+\ u^n\mathcal{S}\big(\,K_2(\xi)\big)\mathcal{S}\left(\left[\sum_{n=0}^{\infty} B_n\right]\right) + u^n\mathcal{S}\big(\,R_2(\xi)\big)\mathcal{S}\left(\left[\sum_{n=0}^{\infty} B_n\right]\right)$$

(3.41)

The recursive relation helps to obtain these results

$$\mathcal{S}[\Upsilon_0(\xi)] =\ \Upsilon(0) + u\Upsilon'(0) + \cdots + u^{n-1}\Upsilon^{n-1}(0) + u^n\mathcal{S}[f_1(\xi)]$$
$$\mathcal{S}[\Upsilon_{k+1}(\xi)] = u^n\mathcal{S}\big(\,K_1(\xi)\big)\mathcal{S}(A_k) + u^n\mathcal{S}\big(\,R_1(\xi)\big)\mathcal{S}\big(\tilde{A}_k\big)$$
$$\mathcal{S}[V_0(\xi)] =\ V(0) + uV'(0) + \cdots + u^{n-1}V^{n-1}(0) + u^n\mathcal{S}[f_2(\xi)]$$
$$\mathcal{S}[V_{k+1}(\xi)] = u^n\mathcal{S}\big(\,K_2(\xi)\big)\mathcal{S}(B_k) + u^n\mathcal{S}\big(\,R_2(\xi)\big)\mathcal{S}(B_k) \qquad (3.42)$$

The combined STAD method for the solution of systems of the Volterra integro-differential equations of the nonlinear type will be clearer by understanding the given below example. [27]

Example 3.4.

Use CSTAD method for solving given Volterra integro-differential equation

$$\Upsilon^{(2)}(\xi) = \frac{7}{3}e^{\xi} - e^{2\xi} - \frac{1}{4}e^{4\xi} + \int_0^{\xi} e^{\xi - t}\big(\Upsilon^2(t) + V^2(t)\big)dt$$

$$V^{(2)}(\xi) = \frac{2}{3}e^{\xi} + 3e^{2\xi} + \frac{1}{3}e^{4\xi} + \int_0^{\xi} e^{\xi - t}\big(\Upsilon^2(t) - V^2(t)\big)dt$$

$$\Upsilon(0) = 1, \Upsilon^{(1)}(0) = 1, V(0) = 1, V^{(1)}(0) = 2$$

(3.43)

Taking Sumudu transforms of both sides of (3.43), we obtain

$$\Upsilon(u) = 1 + u + \frac{7u^2}{3(1-u)} - \frac{u^2}{(1-2u)} - \frac{u^2}{3(1-4u)} + \left(\frac{u^3}{1-u}\right)\,\mathcal{S}[\Upsilon^2(x) + V^2(x)],$$

$$V(u) = 1 + 2u + \frac{2u^2}{3(1-u)} + \frac{3u^2}{(1-2u)} - \frac{u^2}{3(1-4u)} + \left(\frac{u^3}{1-u}\right) \mathcal{S}[\Upsilon^2(\xi) - V^2(\xi)] \tag{3.44}$$

By using (3.39), we have

$$\Upsilon_0(u) = 1 + u + \frac{7u^2}{3(1-u)} - \frac{u^2}{(1-2u)} - \frac{u^2}{3(1-4u)},$$

$$\Upsilon_{k+1}(u) = \left(\frac{u^3}{1-u}\right) \mathcal{S}[A_k(\xi) + B_k(\xi)],$$

$$V_0(u) = 1 + 2u + \frac{2u^2}{3(1-u)} + \frac{3u^2}{(1-2u)} - \frac{u^2}{3(1-4u)}$$

$$V_{k+1}(u) = \left(\frac{u^3}{1-u}\right) \mathcal{S}[A_k(\xi) - B_k(\xi)] \tag{3.45}$$

Recall that Adomian polynomials for $\Upsilon^2(\xi)$ and $V^2(\xi)$ are given by

$$A_O = \Upsilon_0^2, \qquad A_1 = 2\Upsilon_0\Upsilon_{1,}$$

$$A_2 = 2\Upsilon_0\Upsilon_2 + \Upsilon_1^2, \quad A_3 = 2\Upsilon_0\Upsilon_3 + 2\Upsilon_1\Upsilon_{2,}$$

$$B_O = V_0^2, \qquad B_1 = 2V_0V_{1,}$$

$$B_2 = 2V_0V_2 + V_1^2, \quad B_3 = 2V_0V_3 + 2V_1V_{2,} \tag{3.46}$$

the inverse Sumudu transform of (3.44) and with the help of recursive relation equations (3.45), the solution is as follows,

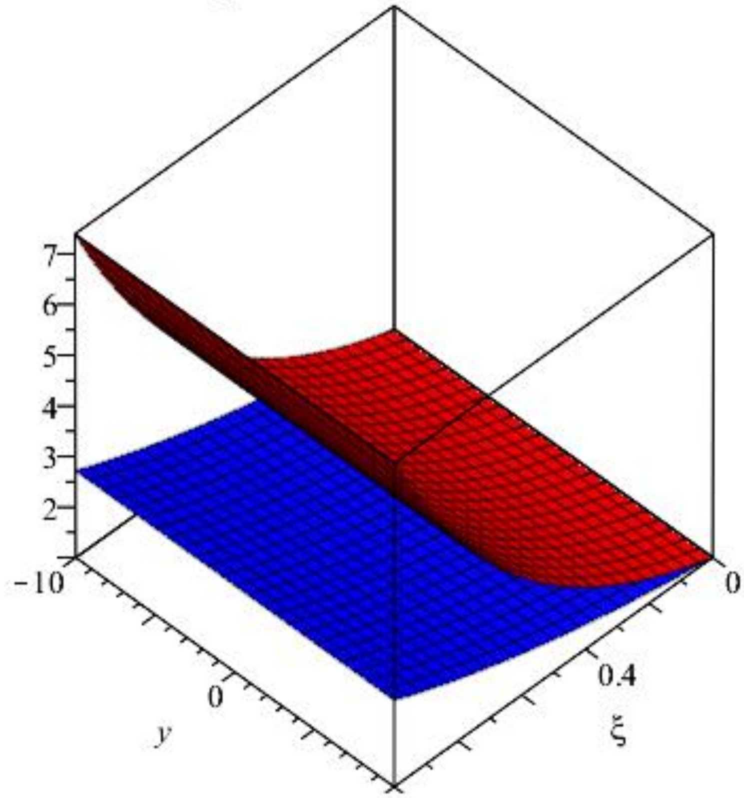
$$\Upsilon(\xi) = \left(1 + \xi + \frac{1}{2!}\xi^2 + \frac{1}{3!}\xi^3 + \frac{1}{4!}\xi^4 + \cdots\right),$$

$$V(\xi) = \left(1 + 2\xi + \frac{1}{2!}(2\xi)^2 + \frac{1}{3!}(2\xi)^3 + \frac{1}{4!}(2\xi)^4 + \cdots\right). \tag{3.47}$$

Then the solution for the above system is given by

$$\Upsilon(\xi) = e^{\xi} \qquad\qquad V(\xi) = e^{2\xi} \tag{3.48}$$

Graphical interpretation of 3.4



Chapter.04

Use of Sumudu Decomposition method for solution of fuzzy Integro-differential equation

4.1 Analysis of method

Consider a Volterra integro-differential equation [13,14]

$$\Upsilon^{n}(\xi,\rho)=f(\xi,\rho)+\int_{0}^{\xi}k(\xi-t)\,\Upsilon(t,\rho)dt \tag{4.1}$$

$$\Upsilon^{k}(0)=\overline{p}=(\underline{p}_{k},\overline{p}_{k});\ _{0\le k\le n-1}$$

By taking sumudu transform on (4.1) of equation

$$\mathcal{S}[\Upsilon^{n}(\xi,\rho)]=\mathcal{S}[f(\xi,\rho)]+\mathcal{S}\left[\int_{0}^{\xi}k(\xi-t)\,\Upsilon(t,\rho)dt\right] \tag{4.2}$$

This will give us,

$$\frac{1}{u^{n}}\mathcal{S}[\Upsilon(\xi,\rho)]-\frac{1}{u^{n}}\Upsilon(0,\rho)-\frac{1}{u^{n-1}}\Upsilon^{(1)}(0,\rho)-\cdots-\frac{\Upsilon^{(n-1)}(0,\rho)}{u}$$
$$=\mathcal{S}[f(\xi,\rho)]+\ \mathcal{S}\left[\int_{0}^{\xi}k(\xi-t)\,\Upsilon(t,\rho)dt\right] \tag{4.3}$$

$$\frac{1}{u^{n}}\mathcal{S}[\underline{\Upsilon}(\xi,\rho)]-\frac{1}{u^{n}}\underline{\Upsilon}(0,\rho)-\frac{1}{u^{n-1}}\underline{\Upsilon}'(0,\rho)-\cdots-\frac{\underline{\Upsilon}^{n-1}(0,\rho)}{u}$$
$$=\mathcal{S}\left[\underline{f}(\xi,\rho)\right]+u\underline{\mathcal{S}[k(\xi-t)]\mathcal{S}[\Upsilon(\xi,t)]} \tag{4.4}$$

$$\frac{1}{u^{n}}\mathcal{S}[\overline{\Upsilon}(\xi,\rho)]-\frac{1}{u^{n}}\overline{\Upsilon}(0,\rho)-\frac{1}{u^{n-1}}\overline{\Upsilon}^{(1)}(0,\rho)-\cdots-\frac{\overline{\Upsilon}^{(n-1)}(0,\rho)}{u}$$
$$=\mathcal{S}[\overline{f}(\xi,\rho)]+u\overline{\mathcal{S}[k(\xi-t)]\mathcal{S}[\Upsilon(\xi,t)]} \tag{4.5}$$

Note that

$$\underline{\Upsilon}(0,\rho)=\underline{p}_{0},\underline{\Upsilon}^{(1)}(0,\rho)=\underline{p}_{1}\ \ldots\ldots\ldots\ \underline{\Upsilon}^{n-1}(0,\rho)=\underline{p}_{n-1}$$

$$\overline{\Upsilon}(0,\rho)=\overline{p}_{0},\Upsilon^{(1)}(0,\rho)=\overline{p}_{1}\ \ldots\ldots\ldots\ \overline{\Upsilon}^{n-1}(0,\rho)=\overline{p}_{n-1}$$

Equation (4) and (5) becomes

$$\frac{1}{u^n}\mathcal{S}[\underline{Y}(\xi,\rho)] - \frac{1}{u^n}\underline{p}_0 - \frac{1}{u^{n-1}}\underline{p}_1 - \cdots - \frac{\underline{p}_{n-1}}{u} = \mathcal{S}\left[\underline{f}(\xi,\rho)\right] + u\mathcal{S}[\underline{k}(\xi-t)]\mathcal{S}[\underline{Y}(\xi,\rho)$$

(4.6)

$$\frac{1}{u^n}\mathcal{S}[\overline{Y}(\xi,\rho)] - \frac{1}{u^n}\overline{p}_0 - \frac{1}{u^{n-1}}\overline{p}_1 - \cdots - \frac{\overline{p}_{n-1}}{u} = \mathcal{S}[\overline{f}(\xi,\rho)] + u\mathcal{S}[\overline{k}(\xi-t)]\mathcal{S}[\overline{Y}(\xi,\rho)]$$

(4.7)

The following cases can be discussed

(i) if $Y(\xi;\rho)$ and $k(\xi;\rho)$ both are positive

$$\mathcal{S}[\underline{k}(\xi,\rho)]\mathcal{S}[\underline{Y}(\xi,\rho)] = \mathcal{S}[\underline{k}(\xi,\rho)]\mathcal{S}[\underline{Y}(\xi,\rho)]$$

$$\mathcal{S}[\overline{k}(\xi,\rho)]\mathcal{S}[\overline{Y}(\xi,\rho)] = \mathcal{S}[\overline{k}(\xi,\rho)]\mathcal{S}[\overline{Y}(\xi,\rho)]$$

(ii) if $Y(\xi;\rho)$ is negative and $k(\xi;\rho)$ is positive

$$\mathcal{S}[\underline{k}(\xi,\rho)]\mathcal{S}[\underline{Y}(\xi,\rho)] = \mathcal{S}[\overline{k}(\xi,\rho)]\mathcal{S}[\underline{Y}(\xi,\rho)]$$

$$\mathcal{S}[\overline{k}(\xi,\rho)]\mathcal{S}[\overline{Y}(\xi,\rho)] = \mathcal{S}[\underline{k}(\xi,\rho)]\mathcal{S}[\overline{Y}(\xi,\rho)]$$

(iii) if $Y(\xi;\rho)$ is positive and $k(\xi;\rho)$ is negative

$$\mathcal{S}[\underline{k}(\xi,\rho)]\mathcal{S}[\underline{Y}(\xi,\rho)] = \mathcal{S}[\underline{k}(\xi,\rho)]\mathcal{S}[\overline{Y}(\xi,\rho)]$$

$$\mathcal{S}[\overline{k}(\xi,\rho)]\mathcal{S}[\overline{Y}(\xi,\rho)] = \mathcal{S}[\overline{k}(\xi,\rho)]\mathcal{S}[\underline{Y}(\xi,\rho)]$$

(iv) if $Y(\xi;\rho)$ and $k(\xi;\rho)$ both are negative

$$\mathcal{S}[\underline{k}(\xi,\rho)]\mathcal{S}[\underline{Y}(\xi,\rho)] = \mathcal{S}[\overline{k}(\xi,\rho)]\mathcal{S}[\overline{Y}(\xi,\rho)]$$

$$\mathcal{S}[\overline{k}(\xi,\rho)]\mathcal{S}[\overline{Y}(\xi,\rho)] = \mathcal{S}[\underline{k}(\xi,\rho)]\mathcal{S}[\underline{Y}(\xi,\rho)]$$

We will explore Case1 and remaining are same

After simplification (6) and (7) becomes

$$\mathcal{S}[\underline{Y}(\xi,\rho)] - \underline{p}_0 - u\underline{p}_1 - \cdots - u^{n-1}\underline{p}_{n-1} = u^n\mathcal{S}\left[\underline{f}(\xi,\rho)\right] + u^{n+1}\,\mathcal{S}[\underline{k}(\xi,\rho)]\mathcal{S}[\underline{Y}(\xi,\rho)]$$

(4.8)

$$\mathcal{S}[\overline{Y}(\xi,\rho)] - \overline{p}_0 - u\overline{p}_1 - \cdots - u^{n-1}\overline{p}_{n-1} = u^n\mathcal{S}[\overline{f}(\xi,\rho)] + u^{n+1}\mathcal{S}[\overline{k}(\xi,\rho)]\mathcal{S}[\overline{Y}(\xi,\rho)]$$

(4.9)

After simplification,

$$\mathcal{S}[\underline{Y}(\xi,\rho)] - u^{n+1}\,\mathcal{S}[\underline{k}(\xi,\rho)]\mathcal{S}[\underline{Y}(\xi,\rho)] = u^n\mathcal{S}\left[\underline{f}(\xi,\rho)\right] + \underline{p}_0 + u\underline{p}_1 + \cdots + u^{n-1}\underline{p}_{n-1}$$

(4.10)

$$\mathcal{S}[Y(\xi,\rho)] - u^{n+1}\mathcal{S}[\overline{k}(\xi,\rho)]\mathcal{S}[\overline{Y}(\xi,\rho)] = u^n\mathcal{S}[\overline{f}(\xi,\rho)] + u\overline{p}_1 + \cdots + u^{n-1}\overline{p}_{n-1}$$

(4.11)

Equation (10) and (11) will give us

$$\mathcal{S}[\underline{Y}(\xi,\rho)] = \frac{u^n \mathcal{S}\left[\underline{f}(\xi,\rho)\right] + \underline{p}_0 + u\underline{p}_1 + \cdots + u^{n-1}\underline{p}_{n-1}}{\left(1 - u^{n+1}\mathcal{S}\left[\underline{k}(\xi - t)\right]\right)}$$

(4.12)

$$\mathcal{S}\left[\overline{Y}(\xi,\rho)\right] = \frac{u^n \mathcal{S}\left[\overline{f}(\xi,\rho)\right] + u\overline{p}_1 + \cdots + u^{n-1}\overline{p}_{n-1}}{(1 - u^{n+1}\mathcal{S}\left[\left(\overline{k}(\xi - t)\right]\right)}$$

(4.13)

By taking the inverse sumudu decomposition method we can get the value of $\overline{Y}(\xi,\rho)$ and $Y(\xi,\rho)$

Now by decomposition method,

$$\sum_{i=0}^{\infty} \underline{Y_i}(\xi,\rho) = \underline{Y_0}(\xi,\rho) + \underline{Y_1}(\xi,\rho) + \underline{Y_2}(\xi,\rho) + \cdots \underline{Y_n}(\xi,\rho)$$

And

$$\sum_{i=0}^{\infty} \overline{Y}_i(\xi,\rho) = \overline{Y}_0(\xi,\rho) + \overline{Y}_1(\xi,\rho) + \overline{Y}_2(\xi,\rho) + \cdots \overline{Y}_n(\xi,\rho)$$

We can write as

Where,

$$\mathcal{S}[\underline{Y_0}(\xi,\rho)] = u^n \mathcal{S}\left[\underline{f}(\xi,\rho)\right] + \underline{p}_0 + u\underline{p}_1 + \cdots + u^{n-1}\underline{p}_{n-1}$$

$$\mathcal{S}\left[\underline{Y_1}(\xi,\rho)\right] = u^{n+1}\underline{\mathcal{S}[k(\xi - t)]\mathcal{S}[Y_0(\xi,\rho)]}$$

$$\mathcal{S}\left[\underline{Y_1}(\xi,\rho)\right] = u^{n+1}\underline{\mathcal{S}[k(\xi - t)]\mathcal{S}[Y_1(\xi,\rho)]}$$

.

.

.

$$\mathcal{S}\left[\underline{Y_n}(\xi,\rho)\right] = u^{n+1}\underline{\mathcal{S}[k(\xi - t)]\mathcal{S}[Y_{n-1}(\xi,\rho)]}$$

Similarly,

$$\mathcal{S}[\overline{Y}_0(\xi,\rho)] = u^n \mathcal{S}\left[\underline{f}(\xi,\rho)\right] + \underline{p}_0 + u\underline{p}_1 + \cdots + u^{n-1}\underline{p}_{n-1}$$

$$\mathcal{S}\left[\overline{Y}_1(\xi,\rho)\right] = u^{n+1}\overline{\mathcal{S}[k(\xi - t)]\mathcal{S}[Y_0(\xi,\rho)]}$$

$$\mathcal{S}\left[\overline{Y}_2(\xi,\rho)\right] = u^{n+1}\overline{\mathcal{S}[k(\xi - t)]\mathcal{S}[Y_1(\xi,\rho)]}$$

.

.

.

$$\mathcal{S}\left[\overline{\Upsilon}_n(\xi,\rho)\right] = u^{n+1}\overline{\mathcal{S}[k(\xi-t)]\mathcal{S}[\Upsilon_{n-1}(\xi,\rho)]}$$

For nonlinear we will use adomian polynomials for nonlinear portion,

$$A_0 = \Upsilon_O{}^2 \quad , \qquad A_1 = 2\Upsilon_O\,\Upsilon_1$$

$$A_2 = 2\Upsilon_O\,\Upsilon_2 + \Upsilon_1{}^2\,,\ \ A_3 = 2\Upsilon_O\,\Upsilon_1 + 2\Upsilon_1\Upsilon_2$$

Then the equation will become,

$$\mathcal{S}\left[\sum_{i=0}^{\infty}\underline{\Upsilon_i}(\xi,\rho)\right] - \underline{p}_0 - u\underline{p}_1 - \cdots - u^{n-1}\underline{p}_{n-1} = u^n\mathcal{S}\left[\underline{f}(\xi,\rho)\right] + u^{n+1}\mathcal{S}\underline{\left[k(x-t)]\mathcal{S}[\sum_{j=1}^{\infty}A_{j-1}\right]} \tag{4.14}$$

$$\mathcal{S}\left[\sum_{i=0}^{\infty}\overline{\Upsilon}_i(\xi,\rho)\right] - \overline{p}_0 - u\overline{p}_1 - \cdots - u^{n-1}\overline{p}_{n-1} = u^n\mathcal{S}\left[\overline{f}(\xi,\rho)\right] + u^{n+1}\mathcal{S}\overline{\left[k(\xi-t)]\mathcal{S}[\sum_{j=1}^{\infty}A_{j-1}\right]}$$

4.2 Numerical Examples

Example 4.1

A linear fuzzy integro-differential Equation is

$$\Upsilon^{(1)}(\xi,\rho) = f(\xi,\rho) - \int_0^{\xi}\Upsilon(t,\rho)dt \tag{4.15}$$

With conditions, $\Upsilon(0,\rho) = (0{,}0)$, where

$$\lambda = 1, 0 \le t \le \xi, 0 \le \rho \le 1, K(\xi,t) = 1,$$

i.e.

$$f(\xi,\rho) = ((\rho^2+\rho),(5-\rho))$$

***Solution*:**

To solve Equation (4.15), for fuzzy integro-differential

$$\begin{cases} \underline{\Upsilon^{(1)}}(\xi,\rho) = f(\xi,\rho) - \int_0^{\xi} \underline{\Upsilon(t,\rho)dt} \\ \overline{\Upsilon^{(1)}}(\xi,\rho) = f(\xi,\rho) - \int_0^{\xi} \overline{\Upsilon(t,\rho)dt} \end{cases} \tag{4.16}$$

$$\begin{cases} \underline{\Upsilon^{(1)}}(\xi,\rho) = (\rho^2+\rho) - \int_0^{\xi} \underline{\Upsilon(t,\rho)dt} \\ \overline{\Upsilon^{(1)}}(\xi,\rho) = (5-\rho) - \int_0^{\xi} \overline{\Upsilon(t,\rho)dt} \end{cases} \tag{4.17}$$

Applying Sumudu transform on (4.17)

$$\begin{cases} \mathcal{S}\left[\underline{\Upsilon^{(1)}}(\xi,\rho)\right] = \mathcal{S}((\rho^2+\rho) - \mathcal{S}\int_0^{\xi} \underline{\Upsilon(t,\rho)dt} \\ \mathcal{S}\overline{[\Upsilon^{(1)}}(\xi,\rho)] = \mathcal{S}((5-\rho) - \mathcal{S}\int_0^{\xi} \overline{\Upsilon(t,\rho)dt} \end{cases} \tag{4.18}$$

This will give us

$$\begin{cases} \frac{1}{u}\mathcal{S}\left(\underline{\Upsilon}(\xi,\rho)\right) - \frac{1}{u}\Upsilon(0,\rho) = (\rho^2+\rho) - u\mathcal{S}\left(\underline{\Upsilon}(\xi,\rho)\right) \\ \frac{1}{u}\mathcal{S}\left(\overline{\Upsilon}(\xi,\rho)\right) - \frac{1}{u}\Upsilon(0,\rho) = (5-\rho) - u\mathcal{S}\left(\overline{\Upsilon}(\xi,\rho)\right) \end{cases} \tag{4.19}$$

On simplification,

$$\begin{cases} \mathcal{S}\left(\underline{\Upsilon}(\xi,\rho)\right) = u(\rho^2+\rho) - u^2\mathcal{S}\left(\underline{\Upsilon}(\xi,\rho)\right) \\ \mathcal{S}\left(\overline{\Upsilon}(\xi,\rho)\right) = u(5-\rho) - u^2\mathcal{S}\left(\overline{\Upsilon}(\xi,\rho)\right) \end{cases} \tag{4.20}$$

Taking inverse transform on two sides of equation (4.20)

$$\begin{cases} \underline{\Upsilon}(\xi,\rho) = \mathcal{S}^{-1}(u(\rho^2+\rho)) - \mathcal{S}^{-1}(u^2\mathcal{S}(\underline{\Upsilon}(\xi,\rho)) \\ \overline{\Upsilon}(\xi,\rho) = \mathcal{S}^{-1}(u(5-\rho)) - \mathcal{S}^{-1}(u^2\mathcal{S}(\overline{\Upsilon}(\xi,\rho)) \end{cases} \tag{4.21}$$

Now Applying Decomposition method for $\underline{\Upsilon}(\xi,\rho)$,

$$\Upsilon_0 = \xi(\alpha^2+\alpha),$$

$$\Upsilon_1 = \mathcal{S}^{-1}(u^2\mathcal{S}\big(\xi(\rho^2+\rho)\big), \qquad \Upsilon_1 = -\frac{\xi^3}{3!}(\rho^2+\rho)$$

$$\Upsilon_2 = \mathcal{S}^{-1}(u^2\mathcal{S}\left(\frac{\xi^3}{3!}(\rho^2+\rho)\right), \qquad \Upsilon_2 = \frac{\xi^5}{5!}(\rho^2+\rho)$$

$$\Upsilon_3 = \mathcal{S}^{-1}(u^2\mathcal{S}\left(\frac{\xi^5}{5!}(\rho^2+\rho)\right), \qquad \Upsilon_3 = -\frac{\xi^7}{7!}(\rho^2+\rho),$$

Similarly, for $\overline{\Upsilon(\xi,\rho)}$

$$\Upsilon_0 = \xi(5-\rho),$$

$$\Upsilon_1 = \mathcal{S}^{-1}(u^2\mathcal{S}\big(\xi(5-\rho)\big), \qquad \Upsilon_1 = -\frac{\xi^3}{3!}(5-\rho),$$

$$\Upsilon_2 = \mathcal{S}^{-1}(u^2\mathcal{S}\left(-\frac{\xi^3}{3!}(5-\rho)\right), \quad \Upsilon_2 = \frac{\xi^5}{5!}(5-\rho),$$

$$\Upsilon_3 = \mathcal{S}^{-1}(u^2\mathcal{S}\left(\frac{\xi^5}{5!}(5-\rho)\right), \qquad \Upsilon_3 = -\frac{\xi^7}{7!}(5-\rho)$$

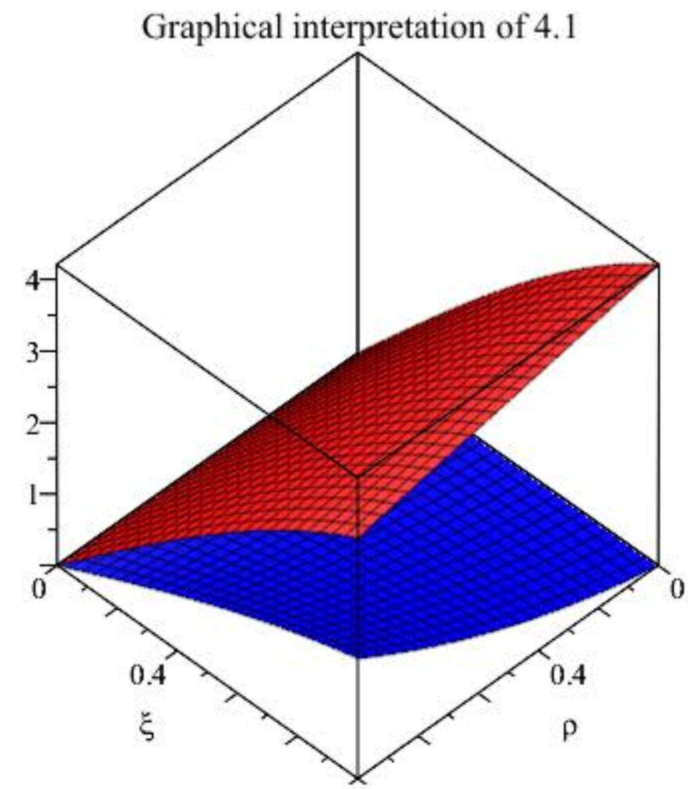
Thus, by utilizing above iterative results the series form solution is given as

$$\begin{cases} \underline{\Upsilon(\xi,\rho)} = \xi(5-\rho) - \frac{\xi^3}{3!}(5-\rho) + \frac{\xi^5}{5!}(5-\rho) - \frac{\xi^7}{7!}(5-\rho) + \ldots. \\ \overline{\Upsilon(\xi,\rho)} = \xi(5-\rho) - \frac{\xi^3}{3!}(5-\rho) + \frac{\xi^5}{5!}(5-\rho) - \frac{\xi^7}{7!}(5-\rho) + \ldots.. \end{cases} \tag{4.22}$$

And the exact solution is given as

$$\begin{cases} \underline{\Upsilon(\xi,\rho)} = sin\xi(\rho^2+\rho) \\ \overline{\Upsilon(\xi,\rho)} = sin\xi(5-\rho) \end{cases} \tag{4.23}$$

The equation number (4.23) is representing the exact solution of (4.15). it can be observed more accurately by its graphical interpretation.



Example.4.2

Consider the following fuzzy Volterra integral-differential equation

$$\begin{cases} \underline{\Upsilon}^{(1)}(\xi,\rho) = (\rho-1) + \int_0^{\xi} \underline{\Upsilon}(t,\rho)dt \\ \overline{\Upsilon}^{(1)}(\xi,\rho) = (1-\rho) + \int_0^{\xi} \overline{\Upsilon}(t,\rho)dt \end{cases}$$

$$\underline{\Upsilon}^{(1)}(0) = 0 = \overline{\Upsilon}^{(1)}(0); 0 \le \rho \le 1, 0 \le t \le \xi, \quad \xi \in [0,1] \tag{4.24}$$

Applying sumudu transform on both sides of the equation (4.24)

$$\begin{cases} \mathcal{S}(\underline{\Upsilon}^{(1)}(\xi,\rho)) = \mathcal{S}((\rho-1)) + \mathcal{S}[\int_0^{\xi} \underline{\Upsilon}(t,\rho)]dt \\ \mathcal{S}(\overline{\Upsilon}^{(1)}(\xi,\rho)) = \mathcal{S}((1-\rho)) + \mathcal{S}[\int_0^{\xi} \overline{\Upsilon}(t,\rho)]dt \end{cases} \tag{4.25}$$

$$\begin{cases} \frac{1}{u}\mathcal{S}(\underline{\Upsilon}\,(\xi,\rho)) - \frac{1}{u}\mathcal{S}(\underline{\Upsilon}\,(0,\rho) = (\rho-1) + u\mathcal{S}[\underline{\Upsilon}(\xi,\rho)] \\ \frac{1}{u}\mathcal{S}(\overline{\Upsilon}\,(\xi,\rho)) - \frac{1}{u}\mathcal{S}(\overline{\Upsilon}\,(0,\rho) = (1-\rho) + u\mathcal{S}[\overline{\Upsilon}(\xi,\rho)] \end{cases}$$

(4.26)

Arranging the equation (4.26)

$$\begin{cases} \mathcal{S}(\underline{\Upsilon}\,(\xi,\rho)) = u(\rho-1) + u^2\mathcal{S}[\underline{\Upsilon}(\xi,\rho)] \\ \mathcal{S}(\overline{\Upsilon}\,(\xi,\rho)) = u(1-\rho) + u^2\mathcal{S}[\overline{\Upsilon}(\xi,\rho)] \end{cases}$$

(4.27)

Applying the inverse sumudu transform in (4.27)

$$\begin{cases} \underline{\Upsilon}\,(\xi,\rho)) = \mathcal{S}^{-1}(u(\rho-1)) + \mathcal{S}^{-1}\left(u^2\mathcal{S}\left[\underline{\Upsilon}(\xi,\rho)\right]\right) \\ \overline{\Upsilon}\,(\xi,\rho)) = \mathcal{S}^{-1}(u(1-\rho)) + \mathcal{S}^{-1}\left(u^2\mathcal{S}\left[\overline{\Upsilon}(\xi,\rho)\right]\right) \end{cases}$$

(4.28)

$$\begin{cases} \underline{\Upsilon}\,(\xi,\rho)) = \mathcal{S}^{-1}(u(\rho-1)) + \mathcal{S}^{-1}\left(u^2\mathcal{S}\left[\underline{\Upsilon}(\xi,\rho)\right]\right) \\ \overline{\Upsilon}\,(\xi,\rho)) = \mathcal{S}^{-1}(u(1-\rho)) + \mathcal{S}^{-1}\left(u^2\mathcal{S}\left[\overline{\Upsilon}(\xi,\rho)\right]\right) \end{cases}$$

Now by decomposition method

$$\underline{\Upsilon}_0(\xi,\rho) = \xi(\rho-1)$$

$$\overline{\Upsilon_0}\,(\xi,\rho)) = \xi(1-\rho)$$

$$\underline{\Upsilon}_1(\xi,\rho) = \mathcal{S}^{-1}\left(u^2\mathcal{S}\left[\underline{\Upsilon_0}(\xi,\rho)\right]\right),\ \ \underline{\Upsilon}_1(\xi,\rho) = \frac{\xi^3}{3!}\ (\rho-1)$$

$$\overline{\Upsilon_1}\,(\xi,\rho)) = \mathcal{S}^{-1}\left(u^2\mathcal{S}\left[\overline{\Upsilon}_0(\xi,\rho)\right]\right),\ \overline{\Upsilon_1}\,(\xi,\rho)) = \frac{\xi^3}{3!}(1-\rho)$$

$$\underline{\Upsilon}_2(\xi,\rho) = \mathcal{S}^{-1}\left(u^2\mathcal{S}\left[\underline{\Upsilon_1}(\xi,\rho)\right]\right),\ \ \underline{\Upsilon}_2(\xi,\rho) = \frac{\xi^5}{3!}\ (\rho-1)$$

$$\overline{\Upsilon_2}\,(\xi,\rho)) = \mathcal{S}^{-1}\left(u^2\mathcal{S}\left[\overline{\Upsilon}_1(\xi,\rho)\right]\right),\ \overline{\Upsilon_2}\,(\xi,\rho)) = \frac{\xi^5}{5!}(1-\rho)$$

$$\underline{\Upsilon}_3(\xi,\rho) = \mathcal{S}^{-1}\left(u^2\mathcal{S}\left[\underline{\Upsilon_2}(\xi,\rho)\right]\right),\ \ \underline{\Upsilon}_3(\xi,\rho) = \frac{\xi^7}{3!}\ (\rho-1)$$

$$\overline{\Upsilon_3}\,(\xi,\rho)) = \mathcal{S}^{-1}\left(u^2\mathcal{S}\left[\overline{\Upsilon}_3(\xi,\rho)\right]\right),\ \overline{\Upsilon_3}\,(\xi,\rho)) = \frac{\xi^7}{5!}(1-\rho)$$

Then the solution in the series form will be

$$\sum_{i=0}^{\infty}\underline{\Upsilon}_i(\xi,\rho) = \underline{\Upsilon}_0(\xi,\rho) + \underline{\Upsilon}_1(\xi,\rho) + \underline{\Upsilon}_2(\xi,\rho) + \underline{\Upsilon}_3(\xi,\rho) + \cdots$$

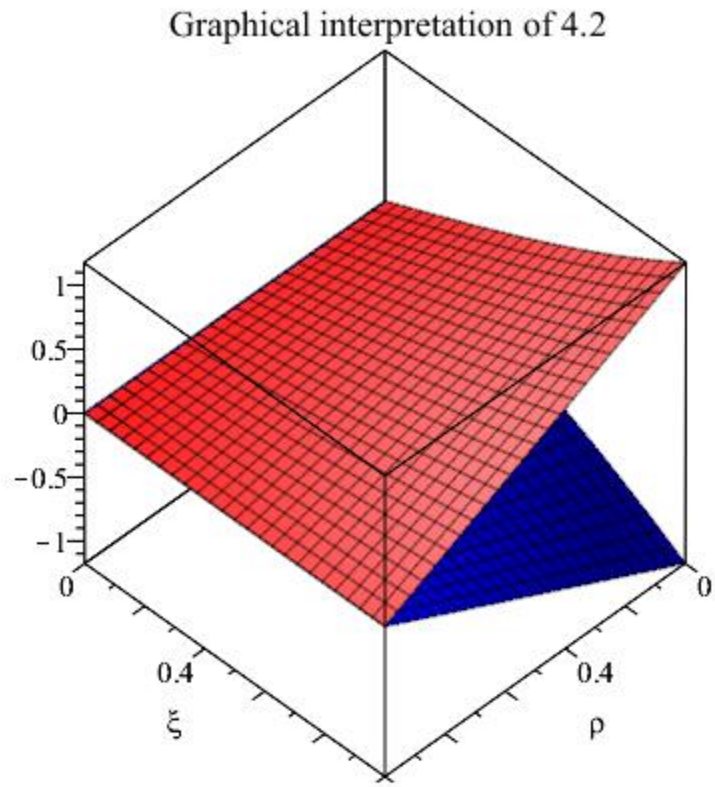
$$= \xi(\rho-1) + \frac{\xi^3}{3!}(\rho-1) + \frac{\xi^5}{3!}(\rho-1) + \frac{\xi^7}{3!}(\rho-1) + \cdots$$

Similarly, for $\overline{Y}(\xi,\rho)$

$$= \xi(1-\rho) + \frac{\xi^3}{3!}(1-\rho) + \frac{\xi^5}{5!}(1-\rho) + \frac{\xi^7}{5!}(1-\rho) + \cdots$$

And the exact solution is given as

$\overline{Y}(\xi,\rho) = (\rho-1)sinht$ and $\overline{Y}(\xi,\rho) = (1-\rho)sinht$



Example.No.4.3

Consider the following fuzzy Volterra integro-differential equation

$$\begin{cases} \underline{Y}^{(1)}(\xi,\rho) = (\rho+1)(1+\xi) + \int_0^{\xi} \underline{Y}(t,\rho)dt \\ \overline{Y}^{(1)}(\xi,\rho) = (\rho-2)(1+\xi) + \int_0^{\xi} \overline{Y}(t,\rho)dt \end{cases}$$

$$\underline{Y}^{(1)}(0) = 0 = \overline{Y}^{(1)}(0); 0 \le \rho \le 1, 0 \le t \le \xi, \quad \xi \in [0,1]$$

(4.29)

Applying sumudu transform on both sides of the equation (4.29)

$$\begin{cases} \mathcal{S}(\underline{Y}^{(1)}(\xi,\rho)) = \mathcal{S}((\rho+1)(1+\xi)) + \mathcal{S}[\int_0^\xi \underline{Y}(t,\rho)]dt \\ \mathcal{S}(\overline{Y}^{(1)}(\xi,\rho)) = \mathcal{S}((\rho-2)(1+\xi)) + \mathcal{S}[\int_0^\xi \overline{Y}(t,\rho)]dt \end{cases}$$

(4.30)

$$\begin{cases} \frac{1}{u}\mathcal{S}(\underline{Y}\,(\xi,\rho)) - \frac{1}{u}\mathcal{S}(\underline{Y}\,(0,\rho) = \mathcal{S}((\rho+1)(1+\xi) + u\mathcal{S}[\underline{Y}(\xi,\rho)] \\ \frac{1}{u}\mathcal{S}(\overline{Y}\,(\xi,\rho)) - \frac{1}{u}\mathcal{S}(\overline{Y}\,(0,\rho) = \mathcal{S}((\rho+1)(1+\xi) + u\mathcal{S}[\overline{Y}(\xi,\rho)] \end{cases}$$

Arranging the equation (4.30)

$$\begin{cases} \mathcal{S}(\underline{Y}\,(\xi,\rho)) = u(\rho+1) + u^2(\rho+1) + u^2\mathcal{S}[\underline{Y}(\xi,\rho)] \\ \mathcal{S}(\overline{Y}\,(\xi,\rho)) = u(\rho-2)+u^2(\rho-2) + u^2\mathcal{S}[\overline{Y}(\xi,\rho)] \end{cases}$$

(4.31)

Taking the inverse sumudu transform of (4.31)

$$\begin{cases} \underline{Y}\,(\xi,\rho)) = \mathcal{S}^{-1}\big(u(\rho+1)\big) + \mathcal{S}^{-1}(u^2(\rho+1) + \mathcal{S}^{-1}\big(u^2\mathcal{S}\big[\underline{Y}(\xi,\rho)\big]\big) \\ \overline{Y}\,(\xi,\rho)) = \mathcal{S}^{-1}\big(u(\rho-2)\big) + \mathcal{S}^{-1}(u^2(\rho-2) + \mathcal{S}^{-1}\big(u^2\mathcal{S}\big[\overline{Y}(\xi,\rho)\big]\big) \end{cases}$$

Now by decomposition method

$$\underline{Y}_0(\xi,\rho) = \xi(\rho+1) + \frac{\xi^2}{2!}(\rho+1)$$

$$\overline{Y_0}\,(\xi,\rho)) = \xi(\rho-2) + \frac{\xi^2}{2!}(\rho-2)$$

$$\underline{Y}_1(\xi,\rho) = \mathcal{S}^{-1}\big(u^2\mathcal{S}\big[\underline{Y_0}(\xi,\rho)\big]\big),\;\; \underline{Y}_1(\xi,\rho) = \frac{\xi^3}{3!}(\rho+1) + \frac{\xi^4}{4!}(\rho+1)$$

$$\overline{Y_1}\,(\xi,\rho)) = \mathcal{S}^{-1}\big(u^2\mathcal{S}\big[\overline{Y}_0(\xi,\rho)\big]\big),\;\; \overline{Y_1}\,(\xi,\rho)) = \frac{\xi^3}{3!}(\rho-2) + \frac{\xi^4}{4!}(\rho-2)$$

$$\underline{Y}_2(\xi,\rho) = \mathcal{S}^{-1}\big(u^2\mathcal{S}\big[\underline{Y_1}(\xi,\rho)\big]\big),\;\; \underline{Y}_2(\xi,\rho) = \frac{\xi^5}{5!}(\rho+1) + \frac{\xi^6}{6!}(\rho+1)$$

$$\overline{Y_2}\,(\xi,\rho)) = \mathcal{S}^{-1}\big(u^2\mathcal{S}\big[\overline{Y}_1(\xi,\rho)\big]\big),\;\; \overline{Y_2}\,(\xi,\rho)) = \frac{\xi^5}{5!}(\rho-2) + \frac{\xi^6}{6!}(\rho-2)$$

$$\underline{Y}_3(\xi,\rho) = \mathcal{S}^{-1}\big(u^2\mathcal{S}\big[\underline{Y_2}(\xi,\rho)\big]\big),\;\; \underline{Y}_3(\xi,\rho) = \frac{\xi^7}{7!}(\rho+1) + \frac{\xi^8}{8!}(\rho+1)$$

$$\overline{Y_3}\,(\xi,\rho)) = \mathcal{S}^{-1}\big(u^2\mathcal{S}\big[\overline{Y}_3(\xi,\rho)\big]\big),\;\; \overline{Y_3}\,(\xi,\rho)) = \frac{\xi^7}{7!}(\rho-2) + \frac{\xi^8}{8!}(\rho-2)$$

Then the solution in the series form will be

$$\sum_{i=0}^{\infty} \underline{\Upsilon}_i(\xi,\rho) = \underline{\Upsilon}_0(\xi,\rho) + \underline{\Upsilon}_1(\xi,\rho) + \underline{\Upsilon}_2(\xi,\rho) + \underline{\Upsilon}_3(\xi,\rho) + \cdots$$

$$= \xi(\rho+1) + \frac{\xi^2}{2!}(\rho+1) + \frac{\xi^3}{3!}(\rho+1) + \frac{\xi^4}{4!}(\rho+1) + \frac{\xi^5}{5!}(\rho+1) + \frac{\xi^6}{6!}(\rho+1)\frac{\xi^7}{7!}(\rho+1)$$

$$+ \frac{\xi^8}{8!}(\rho+1) + \cdots \tag{4.32}$$

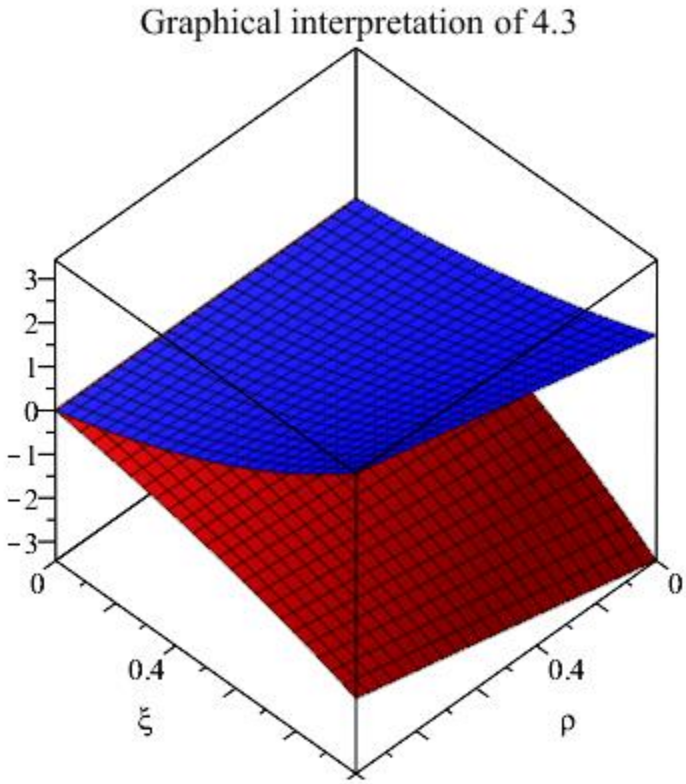
Similarly, for $\overline{\Upsilon}(\xi,\rho)$

$$= \xi(\rho-2) + \frac{\xi^2}{2!}(\rho-2) + \frac{\xi^3}{3!}(\rho-2) + \frac{\xi^4}{4!}(\rho-2) + \frac{\xi^5}{5!}(\rho-2) + \frac{\xi^6}{6!}(\rho-2)\frac{\xi^7}{7!}(\rho-2)$$

$$+ \frac{\xi^8}{8!}(\rho-2) + \cdots \tag{4.33}$$

And the exact solution is given as,

$\underline{\Upsilon}(\xi,\rho) = (\rho+1)(e^x - 1)$ and $\overline{\Upsilon}(\xi,\rho) = (\rho-2)(e^x - 1)$



Example.No.4.4

Consider a Volterra integro-differential equation

$$\Upsilon^{(2)}(\xi,\rho) = f(\xi,\rho) - \int_0^{\xi} (\xi - t)\Upsilon(t,\rho)dt$$

(4.34)

With conditions,

$$u(0,\rho) = (\rho + 1, 3 - \rho); \quad u^{(1)}(0,\rho) = (\rho, 2 - \rho)$$
$$\lambda = 1, 0 \le t \le \xi, 0 \le \rho \le 1, K(\xi,t) = (\xi - t),$$

Solution:

To solve Equation (4.34) for fuzzy-integro differential

$$\begin{cases} \underline{\Upsilon^{(2)}(\xi,\rho)} = f(\xi,\rho) - \int_0^{\xi} (\xi - t)\underline{\Upsilon(t,\rho)dt} \\ \overline{\Upsilon^{(2)}(\xi,\rho)} = f(\xi,\rho) - \int_0^{\xi} (\xi - t)\overline{\Upsilon(t,\rho)dt} \end{cases}$$

(4.35)

$$\begin{cases} \underline{\Upsilon^{(2)}(\xi,\rho)} = (\rho + 2)\xi - \int_0^{\xi} (\xi - t)\underline{\Upsilon(t,\rho)dt} \\ \overline{\Upsilon^{(2)}(\xi,\rho)} = (4 - \rho)\xi - \int_0^{\xi} (\xi - t)\overline{\Upsilon(t,\rho)dt} \end{cases}$$

(4.36)

Applying sumudu transform on both sides of equation (4.36)

$$\begin{cases} \mathcal{S}\left(\underline{\Upsilon^{(2)}(\xi,\rho)}\right) = \mathcal{S}((\rho + 2) - \mathcal{S}\int_0^{\xi} (\xi - t)\underline{\Upsilon(t,\rho)dt} \\ \mathcal{S}(\overline{\Upsilon^{(2)}(\xi,\rho)}) = \mathcal{S}((4 - \rho) - \mathcal{S}\int_0^{\xi} (\xi - t)\overline{\Upsilon(t,\rho)dt} \end{cases}$$

(4.37)

This will give us

$$\begin{cases} \frac{1}{u^2}\mathcal{S}(\underline{\Upsilon(\xi,\rho)} - \frac{1}{u^2}\Upsilon(0,\rho) - \frac{1}{u}\Upsilon^{(1)}(0,\rho) = (\rho^2+\rho) - u\mathcal{S}\int_0^{\xi}(\xi-t)\underline{\Upsilon(t,\rho)dt} \\ \frac{1}{u^2}\mathcal{S}(\overline{\Upsilon(\xi,\rho)} - \frac{1}{u^2}\Upsilon(0,\rho) - \frac{1}{u}\Upsilon^{(1)}(0,\rho) = (5-\rho) - u\mathcal{S}\int_0^{\xi}(\xi-t)\overline{\Upsilon(t,\rho)dt} \end{cases}$$

(4.38)

After simplification,

$$\begin{cases} \mathcal{S}\left(\underline{\Upsilon(\xi,\rho)}\right) = u^3(\rho+2) + u\rho + (\rho+1 + u^4\mathcal{S}\left(\underline{\Upsilon(\xi,\rho)}\right) \\ \mathcal{S}(\overline{\Upsilon(\xi,\rho))} = u^3(4-\rho)) + u(2-\rho) + (3-\rho) + u^4\mathcal{S}(\overline{\Upsilon(\xi,\rho)}) \end{cases}$$

(4.39)

Applying inverse sumudu transform on (4.39)

$$\begin{cases} \underline{\Upsilon(\xi,\rho)} = \mathcal{S}^{-1}(u^3(\rho+2) + \mathcal{S}^{-1}(u\rho) + \mathcal{S}^{-1}(\rho+1) - \mathcal{S}^{-1}(u^4\mathcal{S}\left(\underline{\Upsilon(\xi,\rho)}\right) \\ \overline{\Upsilon(\xi,\rho)} = \mathcal{S}^{-1}(u^3(4-\rho) + \mathcal{S}^{-1}(2-\rho) + \mathcal{S}^{-1}(3-\rho) - \mathcal{S}^{-1}(u^4\mathcal{S}(\overline{\Upsilon(\xi,\rho)})) \end{cases}$$

(4.40)

Now Applying Decomposition method for $\underline{\Upsilon(\xi,\rho)}$, [29]

$\Upsilon_0 = \frac{\xi^3}{3!}(\rho+2) + \xi\rho + (\rho+1)$

$$\Upsilon_1 = \mathcal{S}^{-1}\left(u^4\mathcal{S}(\frac{\xi^3}{3!}(\rho+2) + \xi\rho + (\rho+1))\right), \qquad \Upsilon_1 = \frac{\xi^7}{7!}(\rho+2) + \frac{\xi^5}{5!}\rho + \frac{\xi^4}{4!}(\rho+1)$$

$$\Upsilon_2 = \mathcal{S}^{-1}\left(u^4\mathcal{S}(\frac{\xi^7}{7!}(\rho+2) + \frac{\xi^5}{5!}\rho + \frac{\xi^4}{4!}(\rho+1)\right) \quad \Upsilon_2 = \frac{\xi^{11}}{11!}(\rho+2) + \frac{\xi^9}{9!}\rho + \frac{\xi^8}{8!}(\rho+1)$$

Similarly, we can find $\underline{\Upsilon_3}(\xi,\rho), \underline{\Upsilon_4}(\xi,\rho), \ldots.$

$$\sum_{i=0}^{\infty}\underline{\Upsilon_i}(\xi,\rho) = \underline{\Upsilon_0}(\xi,\rho) + \underline{\Upsilon_1}(\xi,\rho) + \underline{\Upsilon_2}(\xi,\rho) + \ldots.$$

$$= \left(\xi + \frac{\xi^5}{5!} + \frac{\xi^9}{9!} + \cdots..\right)\rho + \left(1 + \frac{\xi^4}{4!} + \frac{\xi^8}{8!} + \cdots..\right)(\rho+1)$$

$$+ \left(\frac{\xi^3}{3!} + \frac{\xi^7}{7!} + \frac{\xi^{11}}{11!}\ldots..\right)(\rho+2) + \cdots$$

(4.41)

Now for $\overline{\Upsilon(\xi,\rho)}$

$$\Upsilon_0 = \frac{\xi^3}{3!}(4-\rho) + \xi(2-\rho) + (3-\rho)$$

$$\Upsilon_1 = \mathcal{S}^{-1}\left(u^4 \mathcal{S}\left(\frac{\xi}{3!}(4-\rho) + \xi(2-\rho) + (3-\rho)\right)\right),$$

$$\Upsilon_1 = \frac{\xi^7}{7!}(4-\rho) + \frac{\xi^5}{5!}(2-\rho) + \frac{\xi^4}{4!}(3-\rho)$$

$$\Upsilon_2 = \mathcal{S}^{-1}\left(u^4 \mathcal{S}\left(\frac{\xi^7}{7!}(4-\rho) + \frac{\xi^5}{5!}(2-\rho) + \frac{\xi^4}{4!}(3-\rho)\right)\right),$$

$$\Upsilon_2 = \frac{\xi^{11}}{11!}(4-\rho) + \frac{\xi^9}{9!}(2-\rho) + \frac{\xi^8}{8!}(3-\rho)$$

Similarly, for $\overline{\Upsilon_i}(\xi,\rho)$

$$\sum_{i=0}^{\infty} \overline{\Upsilon_i}(\xi,\rho) = \overline{\Upsilon_1}(\xi,\rho) + \overline{\Upsilon_2}(\xi,\rho) + \overline{\Upsilon_3}(\xi,\rho) + \ldots.$$

$$= \left(\xi + \frac{\xi^5}{5!} + \frac{\xi^9}{9!} + \cdots..\right)(2-\rho) + \left(1 + \frac{\xi^4}{4!} + \frac{\xi^8}{8!} + \cdots..\right)(3-\rho)$$

$$+ \left(\frac{\xi^3}{3!} + \frac{\xi^7}{7!} + \frac{\xi^{11}}{11!} \ldots..\right)(4-\rho) + \cdots$$

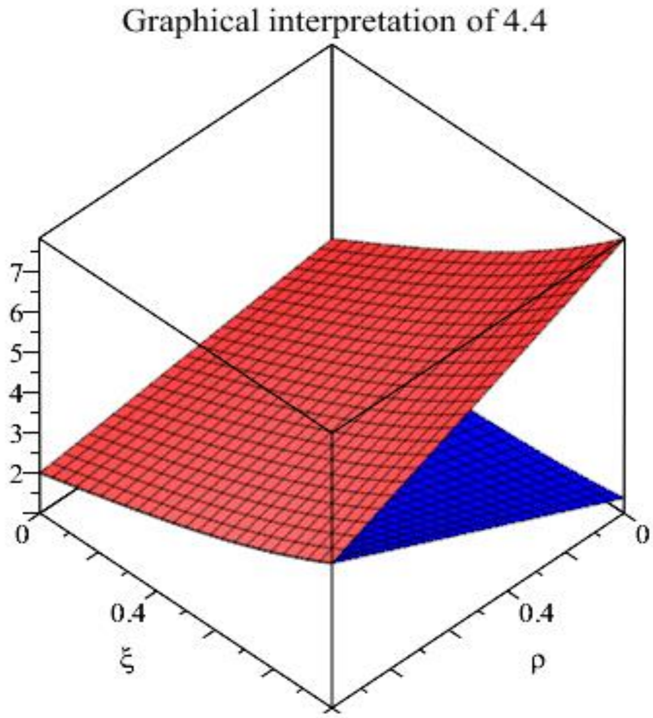
And the exact solution is given as

$$\underline{\Upsilon}(\xi;\rho) = (\rho+2).\frac{1}{2}(sinh\xi - sin\xi) + (\rho+1).\frac{1}{2}(cos\xi + cosh\xi) + (\rho)(sin\xi + sinh\xi)$$

$$\overline{\Upsilon}(\xi;\rho) = (4-\rho).\frac{1}{2}(sinh\xi - sin\xi) + (3-\rho).\frac{1}{2}(cos\xi + cosh\xi) + (2-\rho)(sin\xi + sinh\xi)$$

$$0 \le \rho \le 1 \tag{4.43}$$

The graphical interpretation of the above exact solution is given below which is showing the behavior of our solution. [16,21,26]



Example.4.5

Consider a nonlinear fuzzy Volterra integral differential equation

$$\Upsilon^{(1)}(\xi,\rho) = f(\xi,\rho) - \int_{0}^{\xi} \Upsilon^{2}(t,\rho)dt \tag{4.43}$$

With conditions, $\Upsilon(0,\rho) = (0,0)$, where

$$\lambda = 1, 0 \leq t \leq \xi, 0 \leq \rho \leq 1, K(\xi,t) = 1, \text{i.e}$$
$$f(\xi,\rho) = (\rho, 7-\rho)$$

Solution:

Applying sumudu decomposition on both sides,

To solve Equation (4.43), for fuzzy-integro differential

$$\begin{cases} \underline{\Upsilon^{(1)}(\xi,\rho)} = f(\xi,\rho) - \int_0^{\xi} \underline{\Upsilon^2(t,\rho)dt} \\ \overline{\Upsilon^{(1)}(\xi,\rho)} = f(\xi,\rho) - \int_0^{\xi} \overline{\Upsilon^2(t,\rho)dt} \end{cases} \tag{4.44}$$

$$\begin{cases} \underline{\Upsilon^{(1)}(\xi,\rho)} = \rho - \int_0^{\xi} \underline{\Upsilon^2(t,\rho)dt} \\ \overline{\Upsilon^{(1)}(\xi,\rho)} = 7 - \rho - \int_0^{\xi} \overline{\Upsilon^2(t,\rho)dt} \end{cases} \tag{4.45}$$

Applying sumudu transform on both sides of equation (4.45)

$$\begin{cases} \mathcal{S}(\underline{\Upsilon^{(1)}(\xi,\rho)} = \mathcal{S}(\rho) - \mathcal{S}\int_0^{\xi} \underline{\Upsilon^2(t,\rho)dt} \\ \mathcal{S}(\overline{\Upsilon^{(1)}(\xi,\rho)} = \mathcal{S}(7-\rho) - \mathcal{S}\int_0^{\xi} \overline{\Upsilon^2(t,\rho)dt} \end{cases}$$

(4.46)

This will give us

$$\begin{cases} \frac{1}{u}\mathcal{S}(\underline{\Upsilon(\xi,\rho)} - \frac{1}{u}\Upsilon(0,\rho) = (\rho) - u\mathcal{S}(\underline{\Upsilon^2(\xi,\rho)} \\ \frac{1}{u}\mathcal{S}(\overline{\Upsilon(\xi,\rho)} - \frac{1}{u}\Upsilon(0,\rho) = (7-\rho) - u\mathcal{S}(\overline{\Upsilon^2(\xi,\rho)} \end{cases}$$

(4.47)

$$\begin{cases} \mathcal{S}(\underline{\Upsilon(\xi,\rho)} = u\rho - u^2\mathcal{S}(\Upsilon^2\underline{(\xi,\rho)} \\ \mathcal{S}(\overline{\xi(\xi,\rho)} = u(7-\rho) - u^2\mathcal{S}(\overline{\Upsilon^2(\xi,\rho)} \end{cases}$$

(4.48)

Applying inverse sumudu transform

$$\begin{cases} \underline{\Upsilon(\xi,\rho)} = \mathcal{S}^{-1}(u\rho) - \mathcal{S}^{-1}(u^2\mathcal{S}\left(\Upsilon^2\underline{(\xi,\rho)}\right) \\ \overline{\Upsilon(\xi,\rho)} = \mathcal{S}^{-1}(u(7-\rho)) - \mathcal{S}^{-1}(u^2\mathcal{S}(\overline{\Upsilon^2(\xi,\rho)})) \end{cases}$$

(4.49)

For nonlinear portion,

$$A_0 = \Upsilon_O{}^2 \quad , \qquad A_1 = 2\Upsilon_O\, \Upsilon_1$$

$$A_2 = 2\Upsilon_O\, \Upsilon_2 + \Upsilon_1{}^2\,,\ \ A_3 = 2\Upsilon_O\, \Upsilon_1 + 2\Upsilon_1\Upsilon_2$$

For $\underline{\Upsilon(\xi,\rho)}$

$$\underline{\Upsilon_0}(\xi,\rho) = \xi\rho$$

$$\underline{\Upsilon_1}(\xi,\rho) = \ \mathcal{S}^{-1}(\ u^2\mathcal{S}(k(\xi - t)\mathcal{S}(A_0(\xi))$$

$$\underline{\Upsilon_1}(\xi,\rho) = \ \mathcal{S}^{-1}(\ u^2\mathcal{S}(k(\xi - t)\mathcal{S}((\xi\rho)^2)$$

$$\underline{\Upsilon_1}(\xi,\rho) = \frac{\xi^4}{12}\rho$$

$$\underline{\Upsilon_2}(\xi,\rho) = \ \mathcal{S}^{-1}(\ u^2\mathcal{S}(k(\xi - t)\mathcal{S}(A_1(\xi))$$

$$\underline{\Upsilon_2}(\xi,\rho) = \ \mathcal{S}^{-1}(\ u^2\mathcal{S}(k(\xi - t)\mathcal{S}(2\Upsilon_O\, \Upsilon_1)$$

$$\underline{\Upsilon_2}(\xi,\rho) = \frac{\xi^7}{252}\rho^7$$

Similarly, we can find $\underline{\Upsilon_3}(\xi,\rho), \underline{\Upsilon_4}(\xi,\rho), \dots ..$

$$\sum_{i=0}^{\infty} \underline{\Upsilon_i}(\xi,\rho) = \underline{\Upsilon_0}(\xi,\rho) + \underline{\Upsilon_1}(\xi,\rho) + \underline{\Upsilon_2}(\xi,\rho) + \dots.$$

$$= \xi\rho + \frac{\xi^4}{12}\rho + \frac{\xi^7}{252}\rho^7 + \dots . \tag{4.50}$$

Similarly, for $\overline{\Upsilon_i}(\xi,\rho)$

$$\overline{\Upsilon_0}(\xi,\rho) = \xi(7-\rho)$$

$$\overline{\Upsilon_1}(\xi,\rho) = \mathcal{S}^{-1}(\ u^2\mathcal{S}(k(\xi - t)\mathcal{S}(A_0(\xi)$$

$$\overline{\Upsilon_1}(\xi,\rho) = \mathcal{S}^{-1}(\ u^2\mathcal{S}(k(\xi - t)\mathcal{S}((\xi(7-\rho)^2)$$

$$\overline{\Upsilon_1}(\xi,\rho) = \frac{\xi^4}{12}(7-\rho)^2$$

$$\overline{\Upsilon_2}(\xi,\rho) = \mathcal{S}^{-1}(\ u^2\mathcal{S}(k(\xi - t)\mathcal{S}(A_1(\xi)$$

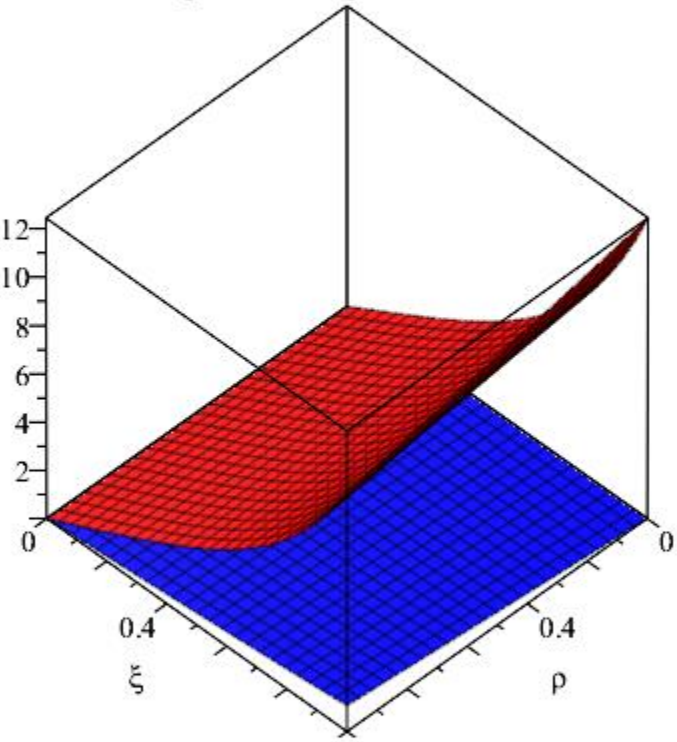
$$\overline{\Upsilon_2}(\xi,\rho) = \mathcal{S}^{-1}(\ u^2\mathcal{S}(k(\xi - t)\mathcal{S}(2\Upsilon_0\Upsilon_1)$$

$$\overline{\Upsilon_1}(\xi,\rho) = \frac{\xi^7}{252}(7-\rho)^3$$

$$\sum_{i=0}^{\infty} \overline{Y_i}(\xi,\rho) = \overline{Y_1}(\xi,\rho) + \overline{Y_2}(\xi,\rho) + \overline{Y_3}(\xi,\rho) + \ldots.$$

$$= \xi(7-\rho) + \frac{\xi^4}{12}(7-\rho)^2 + \frac{\xi^7}{252}(7-\rho)^3 + \ldots \tag{4.51}$$

Graphical interpretation of 4.5



CONCLUSION

Usually it’s difficult to solve fuzzy integro-differential equations analytically. Most probably it’s required to obtain the approximate solutions. In this thesis we developed a numerical technique like sumudu decomposition method for finding the solution of linear and non-linear fuzzy Volterra integro-differential equations. A general method for solving FVIDE is developed. This technique proved reliable and affective from achieved results. It gives fast convergence because by utilizing less number of iterations we get approximate as well as exact solution.

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