A non-stationary binary four point subdivision technique

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بسم الله الرحمن الرحيم

DEDICATED TO

MY PARENTS

Whose prays, encouragement, care,

loving suggestions and moral support

strengthened me to face all hardships

of life and shaped my perception.

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UZMA MUKHTAR

ABSTRACT

A new non-stationary four-point binary approximating subdivision technique with the shape control parameter has been analysed. The proposed technique is the counter part of stationary four-point binary technique [13]. The resulting curve have smoothness C^3 continuous for the wider range of shape control parameter. The role of the shape control parameter has been depicted using the square form of control point.

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Chapter # 1

Introduction

Geometric modelling is the heart of CG and CAGD and covers a wide range of applications. Computer Aided Geometric Design (CAGD) is a branch of applied mathematics that designs smooth curves/ surfaces with algorithms. The CAGD field compiles the visual display. Computer Aided Geometric Design (CAGD) studies the design and handling of curves and surfaces provided by a set of data points in particular. The development of new geometric objects and shapes is an important task in the field of Computer Aided Geometric Design (CAGD), Computer Graphics, Computer Animation Industries and Image processing *etc*. We use a very important subdivision tool to create different geometric objects and shapes in Computer Aided Geometric Design. Algorithms for subdivisions are best suited for computer applications. Subdivision develops different types of smooth curves and surfaces using refining rules by subdividing them from a set of discrete control points.

Because of the development of computer graphics, G. de Rham studied subdivision Scheme (SS) in 1947. Subdivision is a calculation that generates smooth bends and surfaces as a successive arrangement of refined control polygons. The current polygon is included to new points at each refining level and the original points continue to remain or are thrown away across all resultant control polygons. The amount of points placed from level k to level $k+1$ between two successive points is termed also the scheme's arity. If there are 2,3,4 ,…, n inserted then the SS are known as binary, ternary,…, n-ary respectively.

Subdivision is also a technique for construction of smooth curves / surfaces, which first applied as an extension of splines to arbitrary topology control nets. Simplicity and flexibility of the subdivision algorithms make them suitable for many interactive computer graphics applications. Purity of the subdivision lies in the construction of smooth curves / surfaces. However, the uses such as special aspects and animation need generation and construction of composite geometric shapes, which, like real world geometry, carry detail at many scales.

An important characteristic of these schemes is that they are local. There is no need to solve a global equation system. Although the mathematical surface analysis resulting from subdivision algorithms is not always easy. SS takes the attention of scholars and researchers in this field because of its simplicity and ease of understanding. In the last two decades, many papers have been published.

There will be a lot of new CAGD applications in the future. In CAGD, the most frequently used and attractive scheme is SS that draw different types of curves and surfaces. Iterative refinements use the SS to build smooth curves and surfaces from a set of specific CP. SS have been valued in many areas, such as image processing, computer graphics and computer animation, due to their clarity and simplicity. Subdivision systems can be implemented easily and are suitable for computer applications. In general, we can classify subdivision scheme according to the following standards:

i. By the number of control grid edges such as T-grid, Q-grid, H- grid etc.
ii. By topological grid splitting style; i.e., point split and face split.
iii. Limit curve and control polygon like approximation subdivision and interpolatory subdivision by the relationship of limit surface,
iv. By the continuity and smoothness of limit surface for instance C^0 and up to C^m.
v. Uniform subdivision and non-uniform subdivision elements in the same layer.
vi. Like stationary subdivision and dynamic subdivision by the relationship of geometric rule to subdivision layer.
vii. The number of control points inserted between two consecutive points at the level $k+1$ such as; binary, ternary ,…, n-array.

1.1 Subdivision Schemes (SS)

The SS preliminaries are discussed as follows in this section.

1.1.1 Continuous function

A function f is continuous at a point $x = a$, when

- The function f is defined at "a".
- The limit of f as x approaches "a" from the right-hand and left-hand limits exists and is the same.
- The limit of f is equal to f (a) when x approaches "a".

1.1.2 Parameter

A variable on which a collection of various cases is recognized by the number of possible values is known as parameter. Any equation write in the form of parameters is known as parameter equation.

OR

A parameter is a quantity that affects a mathematical object's output or behaviour; but is considered to be constant.

Example

In the set of equations $x = 2\beta + 1$ and $y = \beta^2 + 2$, β is called parameter.

1.1.3 Tension Parameter

Tension parameter is used to control the curve shape.

1.1.4 Iterative

An iterative method is a mathematical procedure that uses an initial guess to generate a sequence of improving an approximate solution for a class of problem in which the nth approximation is derived from the previous ones.

1.1.5 Refinement

A refinement of a cover is a cover such that every element is a subset of an element.

1.1.6 Sequence

A sequence is an arrangement of numbers written in definite order according to some specific rule.

1.1.7 Increasing sequence

Consider a_n is a sequence of nth term, so if $a_{n+1} > a_n$ for all n, the sequence is said to be increasing. This means that for all n, we have $(a_{n+1})(a_n) > 1$.

1.1.8 Decreasing sequence

Consider a_n is a sequence of nth term, so if $a_{n+1} < a_n$ for all n, the sequence is said to be decreasing. This means that for all n, we have $(a_{n+1})(a_n) < 1$.

1.1.9 Control Point

A control point is a member of a set of points used to determine the shape of a spline curve or more generally, a surface or higher dimensional object in computer aided geometric design.

1.1.10 Control Polygon

Control polygon is the sequence of control points in space that is usually used to control an object's shape.

1.1.11 Ratio test

Let $\sum_1^\infty a_n$ be a series of positive terms and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$,

where L is a real number or non-negative numbers

- If $L < 1$, the series $\sum_1^\infty a_n$ converges.
- If $L > 1$, the series $\sum_1^\infty a_n$ diverge.
- If $L = 1$ the test fails to determine convergence or divergence of the series.

1.1.12 Limit curve

In mathematics, a limit is the value approached by a function as the input approaches a specific value.

1.1.13 Asymptotic

Asymptotical is a line approaching a curve, but never touching it. A curve and a line approaching but not intersecting are examples of a curve and a line asymptotic to each other.

1.1.14 Asymptotic equivalent

Asymptotic equivalent is a function whose limit exists and is equal to 1.

1.1.15 Subdivision

A subdivision is a technique of continually refining CP q^0 to produce a pattern on ever increasingly finer polygons $q^0, q^1, q^2, q^3, q^4, \ldots$,thus approaching a polygon limit to a curve.

$$q = \lim_{k\to\infty} q^k.$$

i-e, subdivision characterizes a smooth curve now as the limit of a successive refining sequence.

1.2 Types of SS

Types of SS are:

- Stationary SS and non-stationary SS
- Approximating SS and interpolating SS

1.2.1 Stationary Subdivision Scheme

If in each refining step the mask remains unchanged, it is called a stationary subdivision scheme.

1.2.2 Non-stationary Subdivision Scheme

If in each refining step the mask changes again and again, it is called a non-stationary SS.

1.2.3 Interpolating Subdivision Scheme

For interpolating curve SS, new vertices are calculated and added to the old polygons at each time of subdivision and the limit curve passes through all the vertices of the original CP.

Due to their interpolation property, Interpolation SS are more attractive than approximation schemes in computer aided geometric design. The interpolation subdivisions are also more users friendly.

1.2.4 Approximating Subdivision Scheme

If new points are generated at each refinement level then it is called approximating SS.

1.3 Application of Subdivision Scheme

Computer Aided Geometric Design (CAGD) plays important rolls in providing techniques and algorithms for mathematical description of 2D shapes. CAGD has its influence in many fields like geology and medical science as it has importance in geographic information systems and image processing (IP) respectively. Computer Aided Geometric Design (CAGD) is mostly used in many engineering fields such as aerospace, automotive design (AD), industrial design (ID), Computer Aided Manufacturing (CAM) in numerical analysis, electrical and mechanical engineering. In the emerging era of computer science and engineering, it provides benefits in animation, simulation behaviour, and graphical view of large data and reconstruction of 3D designs form their diagrams and also fitting 3D models for scanned 3D-prints.

Also, Computer graphics provides main ingredient in geometric modelling and analysis which is used in numerical treatment of PDE's. The important applications of CAGD are modelling of 3D shapes in engineering and technology like the shapes of airplanes, ships and cars, controlling and planning surgery of human body, relief maps in cartography for the space objects and drawing machine charts, creating images in the film industries like cartoon, television and advertising, production and quality control, representation products and visualizing of discrete sets of data points.

1.4 Literature Review

Subdivision is an algorithm technique that generates smooth curves and surfaces as a sequence of refined control polygons in succession. Hormann and Sabin [1] introduce the family and determine how the support, the Holder regularity, the accuracy set, the degree of polynomials spanning the limit curves and the behaviour of artefact vary with the parameter identifying the family members. The high order members of that family achieve higher polynomial reproduction degrees. Dyn *et al.* [2] conditions are partly algebraic and easy to check by considering the SS symbol, but also relate to the scheme's parameterization. The four-point ternary interpolatory SS of Cai [3] is analysed with a tension parameter. It is shown that the resulting curve is C^2 for a certain range of the tension parameter.

Hermandez *et al.* [4] shows that the subdivision curve converges and is continuous. In addition, starting with the initial polygon's chord-length parameterization. We get a subdivision curve parameterized by an arc-length multiple. The weights of the masks of the scheme are defined by Daniel and Shummugaraj [5] in terms of some values of trigonometric B-spline functions. Mustafa *et al.* [6] proposed and analysed the m-point approximating SS with single parameter where $m > 1$. Compared to the existing binary and ternary SS, smoothness of schemes is higher.

Zheng and Peng [7] presented an explicit formula that unifies the mask of ternary interpolation and approximation of SS $(2n-1)$ points. Mustafa and Najma [8] generate an approximation property based on local cubic polynomial fitting of the four-point interpolatory curve subdivision. This shows once the scheme is applied to develop a limit curve interpolating irregularly spaced points sampled from a curve in any space dimension with a constrained fourth derivative and the approved parameterization is chordal; the fourth order is the accuracy. Augsdorfer *et al.* [9] showed how subdivision can be divided into stages and how these stages can be manipulated in different ways using the four-point scheme.

Deng and Guozhao, [10] developed an in-centre SS for curve interpolation. Siddiqi and Rehan [11] improved the binary 4-point approximating subdivision scheme by introducing a global tension parameter. A new Subdivision Scheme for corner cutting was also proposed, which generates a limiting curve of C^1 continuity. To examine the order of the derivative countinuity of the two SS, the Laurent polynomial method was used. Siddiqi and Rehan [12] presented ternary three-point approximation of a non-stationary approximation SS that generates a limiting curve of the C^2 family. The proposed scheme may be regarded as a non-stationary counterpart of the stationary ternary three-point scheme. Kim *et al.* [13] proposed a binary subdivision scheme with four points that generates a smooth C^3-continuous limiting curve.

With the help of local cubic polynomial fitting, Floater [14] derived an approximation property of four-point interpolatory curve subdivision. Hao and Renhong [15] derived and investigated 6-point binary subdivision approximating scheme and showed that the scheme is simple and elegant. Sharon and Dyn [16] presented interpolating data consisting of univariate functions by repeated refinements along equidistant parallel lines in a bivariate subdivision scheme. A surface passing through a given set of parametric curves could be practiced by the present method. Pan *et al.* [17] proposed a combined subdivision approximation and interpolation scheme. The relationship between the approximate and the interpolating subdivision is precisely derived from geometric rules operations.

Siddiqi and Younis [18] used the general recursion formula to create an algorithm for m-point binary point approximating SS. Ashraf and Mustafa [19] for even integer > 2 is given a generalized non-stationary subdivision approximating 4-point b-ary. By using Lagrange identities, Mustafa and Bari [20] developed a new family of non-stationary ternary interpolating subdivision schemes. The proposed non-stationary schemes are equivalent to converging stationary schemes asymptotically. Tan *et al.* [21] proposed a new four-point shape preserving C^3 subdivision technique.

A new five-point binary approximating subdivision technique with two parameters is developed by Tan *et al.* [22] to demonstrate curve flexibility. In a case, the five-point scheme is transformed into a four–point scheme that generates continuous limit curves of C^3. Tan *et al.* [23] introduced a new binary SS of five-points with high continuity and preservation of convexity. They showed that the limit curves for the certain range of the parameter are C^k $(k = 0,1, \dots ,7)$. Siddiqi *et al.* [23] four and five points binary non-stationary SS was developed using hyperbolic B-spline basis functions. The main speciality of the hyperbolic SS is that hyperbolas and parabolas can be regenerated quite efficiently. Tan *et al.* [24] developed a non-stationary three-point binary approximating SS that could give a vast range of continuous C^3 smooth curves using the discrete CP.

A new binary non-stationary four-point approximating SS was implemented in this thesis that provides versatility to produce a variation of smooth curves. The smoothness of the SS [24] in chapter 2, has been checked which has maximum derivative continuity C^3. In Chapter 3, smoothness of the proposed scheme has been determined which has maximum derivative continuity C^3. In the end, conclusion has been compiled of the whole thesis.

CHAPTER # 2

A non-stationary binary three-point approximating SS

2.1 Stationary three-point technique

Given the set of zero level CP $q^0 = \{q^0\}_{i \in Z}$, a binary three-point approximating SS for curve design provides a set of new points $\{q_i{}^{k+1}\}_{i \in Z}$ at stage $k+1$ using the following subdivision rules:

$$q_{2i}^{k+1} = \beta_1^k q_{i-1}^k + \beta_2^k q_i^k + \beta_3^k q_{i+1}^k$$

$$q_{2i+1}^{k+1} = \beta_3^k q_{i-1}^k + \beta_2^k q_i^k + \beta_1^k q_{i+1}^k. \qquad \text{……..(2.1)}$$

Where the coefficients $\left\{\beta_j^k\right\}_{j=1,2,3}$ are selected to meet the relationship

$$\beta_1^k + \beta_2^k + \beta_3^k = 1.$$

Hassan and Dodgson [9] introduced the form of a stationary binary technique in which masks $\left\{\beta_j^k\right\}_{j=1,2,3}$ are $\beta_1^k = a, \beta_2^k = 1 - a - b, \beta_3^k = b$.

They found that the scheme is C^1-continuos when $b = \frac{1}{4} + a, -\frac{1}{8} < a < \frac{3}{8}$, C^2-continuos when $b = \frac{1}{4} + a, 0 < a < \frac{1}{8}$ and C^3-continuos when $a = \frac{1}{16}, b = \frac{5}{16}$.

However, the behaviour of the curve is not important to the selection of the values of a and b which meet the conditions of minimum C^2-continuity this means that the generated curves changes in such a small magnitude that there are no significant changes as shown in Figure 2.1.

2.2 A Non-stationary three-point technique

A non-stationary three-point AS technique [24] is defined as in the refining rules (2.1), where the mask $\left\{\beta_i^k\right\}$ are determined as

$$\beta_1^k = g(\beta^{k+`1}), \beta_2^k = \frac{3}{4} - 2g(\beta^{k+`1}), \beta_3^k = g(\beta^{k+`1}) + \frac{1}{4};\ g(\beta^{k+`1}) = \frac{1}{2[(\beta^{k+1})^2 - 1]}, \quad \text{…(2.2)}$$

with

$$\beta^{k+1} = \sqrt{\beta^k + 6}, \beta^0 \in [-6, -5) \cup (-5, +\infty). \qquad \text{…………….(2.3)}$$

In view of the initial parameter $\beta^0 \in [-6, -5) \cup (-5, +\infty)$, the coefficients $\{\beta_i^k\}$ can be calculated by the above formulae at each different levels.

It should be noted that beginning with any $\beta^0 \geq -6$, this gives $\beta^k + 6 \geq 0, \forall k \in Z_{+,}$ so β^{k+1} is well-establish. A vast range of definitions enables to obtain significant variations in the shapes of the curve as shown in Figure 2.1.

2.2.1 Remark 1. From the Figure 2.1, we can see that the curve changes significantly at first as the initial values of β^0 increases in its range and then the curve tends to approximate the defined polygon as $\beta^0 \to +\infty$.

2.2.2 Remark 2. We know from (2.3):

If $\beta^0 = 3$, then $\beta^k = 3$, $\forall\ k \in Z_+$, and the sequence $\{\beta^k\}_{k\in N}$ is stationary and it retrograde the non-stationary SS to the stationary SS.

That is put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 6}$$

$$\beta^1 = \sqrt{3 + 6}$$

$$\beta^1 = \sqrt{9}$$

$$\beta^1 = 3.$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 6}$$

$$\beta^2 = \sqrt{3 + 6}$$

$$\beta^2 = \sqrt{9}$$

$$\beta^2 = 3.$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 6}$$

$$\beta^3 = \sqrt{3 + 6}$$

$$\beta^3 = \sqrt{9}$$

$$\beta^3 = 3.$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 6}$$

$$\beta^4 = \sqrt{3 + 6}$$

$$\beta^4 = \sqrt{9}$$

$$\beta^4 = 3.$$

If $\beta^0 > 3$, then $\beta^k > 3$, the sequence $\{\beta^k\}_{k\in N}$ is decreases strictly and β^k converges to 3 as $k \to \infty$.

Now choose $\beta^0 = 4$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 6}$$

$$\beta^1 = \sqrt{4 + 6}$$

$$\beta^1 = \sqrt{10}$$

$$\beta^1 = 3.162278$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 6}$$

$$\beta^2 = \sqrt{3.162278 + 6}$$

$$\beta^2 = \sqrt{9.162278}$$

$$\beta^2 = 3.02693$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 6}$$

$$\beta^3 = \sqrt{3.02693 + 6}$$

$$\beta^3 = \sqrt{9.02693}$$

$$\beta^3 = 3.00448$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 6}$$

$$\beta^4 = \sqrt{3.00448 + 6}$$

$$\beta^4 = \sqrt{9.00448}$$

$$\beta^4 = 3.00075$$

Now choose $\beta^0 = 5$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 6}$$

$$\beta^1 = \sqrt{5 + 6}$$

$$\beta^1 = \sqrt{11}$$

$$\beta^1 = 3.31662$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 6}$$

$$\beta^2 = \sqrt{3.31662 + 6}$$

$$\beta^2 = \sqrt{9.31662}$$

$$\beta^2 = 3.05231$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 6}$$

$$\beta^3 = \sqrt{3.05231 + 6}$$

$$\beta^3 = \sqrt{9.05231}$$

$$\beta^3 = 3.00871$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 6}$$

$$\beta^4 = \sqrt{3.00871 + 6}$$

$$\beta^4 = \sqrt{9.00871}$$

$$\beta^4 = 3.00145$$

Now choose $\beta^0 = 6$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 6}$$

$$\beta^1 = \sqrt{6 + 6}$$

$$\beta^1 = \sqrt{12}$$

$$\beta^1 = 3.46410$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 6}$$

$$\beta^2 = \sqrt{3.46410 + 6}$$

$$\beta^2 = \sqrt{9.46410}$$

$$\beta^2 = 3.07638$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 6}$$

$$\beta^3 = \sqrt{3.07638 + 6}$$

$$\beta^3 = \sqrt{9.07638}$$

$$\beta^3 = 3.01270$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 6}$$

$$\beta^4 = \sqrt{3.01270 + 6}$$

$$\beta^4 = \sqrt{9.01270}$$

$$\beta^4 = 3.00212$$

Similarly we can check for $\beta^0 = 7$ and so on.

If $\beta^0 < 3$, then $\beta^k < 3$, the sequence $\{\beta^k\}_{k\in N}$ is increases strictly, and β^k converges to 3 as $k \to \infty$.

Choose $\beta^0 = 2$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 6}$$

$$\beta^1 = \sqrt{2 + 6}$$

$$\beta^1 = \sqrt{8}$$

$$\beta^1 = 2.828427$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 6}$$

$$\beta^2 = \sqrt{2.828427 + 6}$$

$$\beta^2 = \sqrt{8.828427}$$

$$\beta^2 = 2.971267$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 6}$$

$$\beta^3 = \sqrt{2.971267 + 6}$$

$$\beta^3 = \sqrt{8.971267}$$

$$\beta^3 = 2.995207$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 6}$$

$$\beta^4 = \sqrt{2.995207 + 6}$$

$$\beta^4 = \sqrt{8.995207}$$

$$\beta^4 = 2.99920106$$

Now choose $\beta^0 = 1$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 6}$$

$$\beta^1 = \sqrt{1 + 6}$$

$$\beta^1 = \sqrt{7}$$

$$\beta^1 = 2.64575$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 6}$$

$$\beta^2 = \sqrt{2.64575 + 6}$$

$$\beta^2 = \sqrt{8.64575}$$

$$\beta^2 = 2.940366$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 6}$$

$$\beta^3 = \sqrt{2.940366 + 6}$$

$$\beta^3 = \sqrt{8.940366}$$

$$\beta^3 = 2.990044$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 6}$$

$$\beta^4 = \sqrt{2.990044 + 6}$$

$$\beta^4 = \sqrt{8.990044}$$

$$\beta^4 = 2.998340$$

Now choose $\beta^0 = 0$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 6}$$

$$\beta^1 = \sqrt{0 + 6}$$

$$\beta^1 = \sqrt{6}$$

$$\beta^1 = 2.4495$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 6}$$

$$\beta^2 = \sqrt{2.4495 + 6}$$

$$\beta^2 = \sqrt{8.4495}$$

$$\beta^2 = 2.906802$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 6}$$

$$\beta^3 = \sqrt{2.906802 + 6}$$

$$\beta^3 = \sqrt{8.906802}$$

$$\beta^3 = 2.984426$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 6}$$

$$\beta^4 = \sqrt{2.984426 + 6}$$

$$\beta^4 = \sqrt{8.984426}$$

$$\beta^4 = 2.99740$$

Similarly, we can check for $\beta^0 = -1$ and so on.

Therefore, in the definition domain of any β^0, we always have

$$\lim_{k \to +\infty} \beta^k = 3.$$

2.3 Smoothness Analysis

In this section, we want to show that, in view of the initial polygon, the SS proposed in section 2.2 gives a continuous smooth curve of C^3 for any selection of the initial parameter value β^0 in its definition. To show this, we quote Dyn and Levin's well-known results [25], which relate to the smoothness of a non-stationary technique with its asymptotically equivalent case stationary counterpart.

2.3.1 Theorem. A non-stationary SS defined by the masks in (2.2) is asymptotically equivalent to the case stationary technique with masks in (2.1) with $\beta_1^k = \frac{1}{16}, \beta_2^k = \frac{5}{8}, \beta_3^k = \frac{5}{16}$. C^3-continuous limit curves are therefore generated.

Proof. In order to prove that the non-stationary scheme converges to a C^3-continuous limit curves, its second divided difference mask should be obtained. The scheme mask is

$$m^k = \left[g(\beta^{k+1}), \frac{1}{4} + g(\beta^{k+1}), \frac{3}{4} - 2g(\beta^{k+1}), \frac{3}{4} - 2g(\beta^{k+1}), \frac{1}{4} + g(\beta^{k+1}), g(\beta^{k+1})\right]$$

Then it turns out that his 1st divided difference masks are

$$e_{(1)}^k = 2\left[g(\beta^{k+1}), \frac{1}{4}, \frac{1}{2} - 2g(\beta^{k+1}), \frac{1}{4}, g(\beta^{k+1})\right]$$

Then it turns out that his 2nd divided difference masks are

$$e_{(2)}^k = 4\left[g(\beta^{k+1}), \frac{1}{4} - g(\beta^{k+1}), \frac{1}{4} - g(\beta^{k+1}), g(\beta^{k+1})\right]$$

Then it turns out that his 3rd divided difference masks are

$$e_{(3)}^k = 8\left[g(\beta^{k+1}), \frac{1}{4} - 2g(\beta^{k+1}), g(\beta^{k+1})\right]$$

Now, the application of Remark 2

$$e_{(3)}^{\infty} = \lim_{k\to+\infty} e_3^k = 8\left[\frac{1}{16}, \frac{1}{8}, \frac{1}{16}\right]$$

Which is only the mask of the third divided differences of the stationary scheme with coefficients in (2.1) with $\beta_1^k = \frac{1}{16}, \beta_2^k = \frac{5}{8}, \beta_3^k = \frac{5}{16}$, and in [9], Hassan and Dodgson proved that in this case the stationary scheme is C^3-continuous, the scheme associated with e_3^{∞} is C^3. Now if it's

$$\sum_{k=0}^{+\infty} \left\| e_{(3)}^k - e_3^{\infty} \right\|_{\infty} < +\infty \qquad \text{……… (2.4)}$$

The two schemes are then equivalent asymptotically. And we can conclude that the e_3^{∞} scheme is C[3].

Since $\quad e_{(3)}^{k} - e_3^{\infty} = 8[g(\beta^{k+1}) - \frac{1}{16}, 2(\frac{1}{16} - g(\beta^{k+1})),\ g(\beta^{k+1}) - \frac{1}{16}]$

$$\left\|e_{(3)}^{k} - e_3^{\infty}\right\|_{\infty} = 8max\left\{2\left|g(\beta^{k+1}) - \frac{1}{16}\right|, 2\left|\frac{1}{16} - g(\beta^{k+1})\right|\right\}$$

$$\left\|e_{(3)}^{k} - e_3^{\infty}\right\|_{\infty} = 16\left|g(\beta^{k+1}) - \frac{1}{16}\right|$$

To prove (2.4), we must prove the smoothness between the series

$$\sum_{k=0}^{+\infty}\left|g(\beta^{k+1}) - \frac{1}{16}\right| \qquad \ldots\ldots\ldots\ldots \ (2.5)$$

which depends on the $g(\beta^{k+1})$ function. Now, since $g(\beta^{k+1})$ is expressed in relation (2.2) in terms of the parameter β^{k+1}, we will study the behaviour of (2.5), since β^{k+1} varies in the interval [0, +∞). From now on

$\boldsymbol{g(\beta^{k+1}) - \frac{1}{16} = 0 \Leftrightarrow \beta^{k+1} = 3, (i.e., \beta^{k} = 3)}$

$= \frac{1}{2[(\beta^{k+1})^2-1]} - \frac{1}{16}$

$= \frac{1}{2[(3)^2-1]} - \frac{1}{16}$

$= \frac{1}{2(8)} - \frac{1}{16}$

$= \frac{1}{16} - \frac{1}{16}$

$= 0$

$\boldsymbol{g(\beta^{k+1}) - \frac{1}{16} > 0 \Leftrightarrow \beta^{k+1} \in (1, 3)}$**(i.e.,** $\boldsymbol{\beta^{k} \in [-4, 3)}$)

$= \frac{1}{2[(\beta^{k+1})^2-1]} - \frac{1}{16}$

For $k = 0$

$= \frac{1}{2[(\beta^{1})^2-1]} - \frac{1}{16}$

Now choose $\boldsymbol{\beta^0 = -4, (i.e., \beta^1 = \sqrt{2})}$

$= \frac{1}{2[(\sqrt{2})^2-1]} - \frac{1}{16}$

$= \frac{1}{2(1)} - \frac{1}{16}$

$= \frac{1}{2} - \frac{1}{16}$

$= 0.5 - 0.0625$

$= 0.4375 > 0.$

For $k = 1$

$= \frac{1}{2[(\beta^2)^2 - 1]} - \frac{1}{16}$

$\beta^1 = 1.4142, (i.e., \beta^2 = 2.7229)$

$= \frac{1}{2[(2.7229)^2 - 1]} - \frac{1}{16}$

$= \frac{1}{12.8284} - \frac{1}{16}$

$= 0.07795 - 0.0625$

$= 0.01545 > 0.$

For $k = 2$

$= \frac{1}{2[(\beta^3)^2 - 1]} - \frac{1}{16}$

$\beta^2 = 2.7229, (i.e., \beta^3 = 2.9535)$

$= \frac{1}{2[(2.9535)^2 - 1]} - \frac{1}{16}$

$= \frac{1}{15.4463} - \frac{1}{16}$

$= 0.06474 - 0.0625$

$= 0.00224 > 0.$

For $k = 3$

$= \frac{1}{2[(\beta^4)^2 - 1]} - \frac{1}{16}$

$\beta^3 = 2.9535, (i.e., \beta^4 = 2.9922)$

$= \frac{1}{2[(2.9922)^2 - 1]} - \frac{1}{16}$

$= \frac{1}{15.9065} - \frac{1}{16}$

$= 0.0629 - 0.0625$

$= 0.0004 > 0.$

Again for $k = 0$

$= \frac{1}{2[(\beta^1)^2-1]} - \frac{1}{16}$

Now choose $\boldsymbol{\beta^0 = -2, (i.e., \beta^1 = 2)}$

$= \frac{1}{2[(2)^2-1]} - \frac{1}{16}$

$= \frac{1}{2(3)} - \frac{1}{16}$

$= \frac{1}{6} - \frac{1}{16}$

$= 0.16666 - 0.0625$

$= 0.1041660 > 0.$

For $k = 1$

$= \frac{1}{2[(\beta^2)^2-1]} - \frac{1}{16}$

$\beta^1 = 2, (i.e., \beta^2 = 2.82843)$

$= \frac{1}{2[(2.82843)^2-1]} - \frac{1}{16}$

$= \frac{1}{14.000} - \frac{1}{16}$

$= 0.07143 - 0.0625$

$= 0.008929 > 0.$

For $k = 2$

$= \frac{1}{2[(\beta^3)^2-1]} - \frac{1}{16}$

$\beta^2 = 2.82843, (i.e., \beta^3 = 2.97127)$

$= \frac{1}{2[(2.97127)^2-1]} - \frac{1}{16}$

$= \frac{1}{15.6569} - \frac{1}{16}$

$= 0.06387 - 0.0625$

$= 0.00137 > 0.$

For $k = 3$

$$= \frac{1}{2[(\beta^4)^2 - 1]} - \frac{1}{16}$$

$\beta^3 = 2.97127, (i.e., \beta^4 = 2.99521)$

$$= \frac{1}{2[(2.99521)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{15.94257} - \frac{1}{16}$$

$= 0.06273 - 0.0625$

$= 0.00023 > 0.$

Again for $k = 0$

$$= \frac{1}{2[(\beta^1)^2 - 1]} - \frac{1}{16}$$

Now choose $\boldsymbol{\beta^0 = -1}, (\boldsymbol{i.e., \beta^1 = \sqrt{5}})$

$$= \frac{1}{2[(\sqrt{5})^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{8} - \frac{1}{16}$$

$= 0.125 - 0.0625$

$= 0.0625 > 0.$

For $k = 1$

$$= \frac{1}{2[(\beta^2)^2 - 1]} - \frac{1}{16}$$

$\beta^1 = 2.236, (i.e., \beta^2 = 2.86984)$

$$= \frac{1}{2[(2.86984)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{14.47196} - \frac{1}{16}$$

$= 0.069099 - 0.0625$

$= 0.006599 > 0.$

For $k = 2$

$= \frac{1}{2[(\beta^3)^2-1]} - \frac{1}{16}$

$\beta^2 = 2.86984, (i.e., \beta^3 = 2.97823)$

$= \frac{1}{2[(2.97823)^2-1]} - \frac{1}{16}$

$= \frac{1}{15.7397} - \frac{1}{16}$

$= 0.063534 - 0.0625$

$= 0.001034 > 0.$

For $k = 3$

$= \frac{1}{2[(\beta^4)^2-1]} - \frac{1}{16}$

$\beta^3 = 2.97823, (i.e., \beta^4 = 2.99637)$

$= \frac{1}{2[(2.99637)^2-1]} - \frac{1}{16}$

$= \frac{1}{15.9565} - \frac{1}{16}$

$= 0.06267 - 0.0625$

$= 0.00017 > 0.$

$\boldsymbol{g(\beta^{k+1}) - \frac{1}{16} < 0 \Leftrightarrow \beta^{k+1}[0, 1) \cup (3, +\infty)}$**(i.e.,** $\boldsymbol{\beta^k \in [-6, -5) \cup (3, +\infty)}$**)**

Again for $k = 0$

$= \frac{1}{2[(\beta^1)^2-1]} - \frac{1}{16}$

Now choose $\boldsymbol{\beta^0 = 4, (i.e., \beta^1 = \sqrt{10})}$

$= \frac{1}{2[(\sqrt{10})^2-1]} - \frac{1}{16}$

$= \frac{1}{2(9)} - \frac{1}{16}$

$= \frac{1}{18} - \frac{1}{16}$

$= 0.05555 - 0.0625$

$= -0.00695 < 0.$

For $k = 1$

$= \frac{1}{2[(\beta^2)^2-1]} - \frac{1}{16}$

$\beta^1 = 3.1623, (i.e., \beta^2 = 3.0269)$

$= \frac{1}{2[(3.0269)^2-1]} - \frac{1}{16}$

$= \frac{1}{16.3242} - \frac{1}{16}$

$= 0.06126 - 0.0625$

$= -0.00124 < 0.$

For $k = 2$

$= \frac{1}{2[(\beta^3)^2-1]} - \frac{1}{16}$

$\beta^2 = 3.0269, (i.e., \beta^3 = 3.0045)$

$= \frac{1}{2[(3.0045)^2-1]} - \frac{1}{16}$

$= \frac{1}{16.0540} - \frac{1}{16}$

$= 0.06229 - 0.0625$

$= -0.00021 < 0.$

For $k = 3$

$= \frac{1}{2[(\beta^4)^2-1]} - \frac{1}{16}$

$\beta^3 = 3.0045, (i.e., \beta^4 = 3.00075)$

$= \frac{1}{2[(3.00075)^2-1]} - \frac{1}{16}$

$= \frac{1}{16.00900} - \frac{1}{16}$

$= 0.06246 - 0.0625$

$= -0.00004 < 0.$

Again for $k = 0$

$$= \frac{1}{2[(\beta^1)^2 - 1]} - \frac{1}{16}$$

Now choose $\boldsymbol{\beta^0 = 5}$, ($\boldsymbol{i.e., \beta^1 = \sqrt{11} = 3.3166}$)

$$= \frac{1}{2[(\sqrt{11})^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{20} - \frac{1}{16}$$

$$= 0.05 - 0.0625$$

$$= -0.0125 < 0.$$

For $k = 1$

$$= \frac{1}{2[(\beta^2)^2 - 1]} - \frac{1}{16}$$

$\beta^1 = 3.3166, (i.e., \beta^2 = 3.0523)$

$$= \frac{1}{2[(3.0523)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{16.6331} - \frac{1}{16}$$

$$= 0.060121 - 0.0625$$

$$= -0.002379 < 0\ .$$

For $k = 2$

$$= \frac{1}{2[(\beta^3)^2 - 1]} - \frac{1}{16}$$

$\beta^2 = 3.0523, (i.e., \beta^3 = 3.00870)$

$$= \frac{1}{2[(3.00870)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{16.1046} - \frac{1}{16}$$

$$= 0.06209 - 0.0625$$

$$= -0.00041 < 0.$$

For $k = 3$

$= \frac{1}{2[(\beta^4)^2-1]} - \frac{1}{16}$

$\beta^3 = 3.00870, (i.e., \beta^4 = 3.00145)$

$= \frac{1}{2[(3.00145)^2-1]} - \frac{1}{16}$

$= \frac{1}{16.0174} - \frac{1}{16}$

$= 0.06243 - 0.0625$

$= -0.00007 < 0.$

Again for $k = 0$

$= \frac{1}{2[(\beta^1)^2-1]} - \frac{1}{16}$

Now choose $\boldsymbol{\beta^0 = 10, (i.e., \beta^1 = 4)}$

$= \frac{1}{2[(4)^2-1]} - \frac{1}{16}$

$= \frac{1}{30} - \frac{1}{16}$

$= 0.03333 - 0.0625$

$= -0.02917 < 0.$

For $k = 1$

$= \frac{1}{2[(\beta^2)^2-1]} - \frac{1}{16}$

$\beta^1 = 4, (i.e., \beta^2 = \sqrt{10})$

$= \frac{1}{2[(\sqrt{10})^2-1]} - \frac{1}{16}$

$= \frac{1}{2(9)} - \frac{1}{16}$

$= \frac{1}{18} - \frac{1}{16}$

$= 0.05555 - 0.0625$

$= -0.006944 < 0.$

For $k = 2$

$$= \frac{1}{2[(\beta^3)^2-1]} - \frac{1}{16}$$

$\beta^2 = 3.1623, (i.e., \beta^3 = 3.0269)$

$$= \frac{1}{2[(3.0269)^2-1]} - \frac{1}{16}$$

$$= \frac{1}{16.3242} - \frac{1}{16}$$

$$= 0.06126 - 0.0625$$

$$= -0.00124 < 0.$$

For $k = 3$

$$= \frac{1}{2[(\beta^4)^2-1]} - \frac{1}{16}$$

$\beta^3 = 3.0269, (i.e., \beta^4 = 3.0045)$

$$= \frac{1}{2[(3.0045)^2-1]} - \frac{1}{16}$$

$$= \frac{1}{16.0540} - \frac{1}{16}$$

$$= 0.06229 - 0.0625$$

$$= -0.00021 < 0.$$

We are therefore discussing the smoothness of (2.5) according to the three cases:

2.3.2 Case 1:

$\boldsymbol{\beta^{k+1} = 3 \left(i.e., \beta^k = 3\right)}$.

Then $\left\|e^k_{(3)} - e^\infty_3\right\|_\infty = 16\left[g(\beta^{k+1}) - \frac{1}{16}\right]$

$$\left\|e^k_{(3)} - e^\infty_3\right\|_\infty = 16\left[\frac{1}{2}\frac{1}{\left[\left(\beta^{k+1}\right)^2-1\right]} - \frac{1}{16}\right]$$

$$= 16\left[\frac{1}{2[(3)^2-1]} - \frac{1}{16}\right]$$

$$= 16\left[\frac{1}{16} - \frac{1}{16}\right]$$

$$= 0$$

Smoothness of (2.5) follows.

2.3.3 Case 2:

$\boldsymbol{\beta^0} \in (\boldsymbol{-5}, \boldsymbol{3})(\boldsymbol{i.e.}, \boldsymbol{\beta^{k+1}} \in (\boldsymbol{1}, \boldsymbol{3}))$.

In this case

$$\left\| e_{(3)}^k - e_3^\infty \right\|_\infty = 16 \left[g(\beta^{k+1}) - \frac{1}{16} \right] \backslash$$

$$g(\beta^{k+1}) - \frac{1}{16} = \frac{1}{2} \frac{1}{\left[\left(\beta^{k+1}\right)^2 - 1 \right]} - \frac{1}{16}$$

For $k = 0$.

$$g(\beta^1) - \frac{1}{16} = \frac{1}{2} \frac{1}{[(\beta^1)^2 - 1]} - \frac{1}{16}$$

Choose $\boldsymbol{\beta^0} = \boldsymbol{-4}, (\boldsymbol{i.e.}, \boldsymbol{\beta^1} = \sqrt{\boldsymbol{2}})$.

$$g(\beta^1) - \frac{1}{16} = \frac{1}{2} \frac{1}{\left[\left(\sqrt{2}\right)^2 - 1 \right]} - \frac{1}{16}$$

$$= \frac{1}{2} - \frac{1}{16}$$

$$= 0.5 - 0.0625$$

$$= 0.4375 < +\infty.$$

For $k = 1$

$$g(\beta^2) - \frac{1}{16} = \frac{1}{2[(\beta^2)^2 - 1]} - \frac{1}{16}$$

$\beta^1 = 1.4142, (i.e., \beta^2 = 2.7229)$.

$$= \frac{1}{2[(2.7229)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{12.8284} - \frac{1}{16}$$

$$= 0.07795 - 0.0625$$

$$= 0.01545 < +\infty.$$

For $k = 2$

$$g(\beta^3) - \frac{1}{16} = \frac{1}{2[(\beta^3)^2 - 1]} - \frac{1}{16}$$

$\beta^2 = 2.7229, (i.e., \beta^3 = 2.9535)$.

$$= \frac{1}{2[(2.7229)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{12.8284} - \frac{1}{16}$$

$$= 0.07795 - 0.0625$$

$$= 0.01545 < +\infty.$$

For $k = 3$

$$g(\beta^3) - \frac{1}{16} = \frac{1}{2[(\beta^4)^2 - 1]} - \frac{1}{16}$$

$\beta^3 = 2.9535, (i.e., \beta^4 = 2.9922).$

$$= \frac{1}{2[(2.9922)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{15.9065} - \frac{1}{16}$$

$$= 0.0629 - 0.0625$$

$$= 0.0004 < +\infty.$$

Again for $k = 0$

$$g(\beta^1) - \frac{1}{16} = \frac{1}{2} \frac{1}{[(\beta^1)^2 - 1]} - \frac{1}{16}$$

Choose $\boldsymbol{\beta^0 = -2}, (\boldsymbol{i.e.}, \boldsymbol{\beta^1 = 2}).$

$$= \frac{1}{2} \frac{1}{[(2)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{6} - \frac{1}{16}$$

$$= 0.16666 - 0.0625$$

$$= 0.1041660 < +\infty.$$

For $k = 1$

$$g(\beta^2) - \frac{1}{16} = \frac{1}{2[(\beta^2)^2 - 1]} - \frac{1}{16}$$

$\beta^1 = 2, (i.e., \beta^2 = 2.82843).$

$$= \frac{1}{2[(2.82843)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{14.000} - \frac{1}{16}$$

$$= 0.07143 - 0.0625$$

$$= 0.008929 < +\infty.$$

For $k = 2$

$$g(\beta^3) - \frac{1}{16} = \frac{1}{2[(\beta^3)^2 - 1]} - \frac{1}{16}$$

$\beta^2 = 2.82843, (i.e., \beta^3 = 2.97127).$

$$= \frac{1}{2[(\ 2.97127\ )^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{15.6569} - \frac{1}{16}$$

$$= 0.06387 - 0.0625$$

$$= 0.00137 < +\infty.$$

For $k = 3$

$$g(\beta^4) - \frac{1}{16} = \frac{1}{2[(\beta^4)^2 - 1]} - \frac{1}{16}$$

$\beta^3 = 2.97127, (i.e., \beta^4 = 2.99521).$

$$g(\beta^4) - \frac{1}{16} = \frac{1}{2[(\ 2.99521\ )^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{15.94257} - \frac{1}{16}$$

$$= 0.06273 - 0.0625$$

$$= 0.00023 < +\infty.$$

Thus

$$\sum_{k=0}^{+\infty} \left(g(\ \beta^{k+1}) - \frac{1}{16} \right) = \sum_{k=0}^{+\infty} \left(\frac{1}{2} \frac{1}{\left[\left(\beta^{k+1} \right)^2 - 1 \right]} - \frac{1}{16} \right) < +\infty.$$

As $\beta^{k+1} = \sqrt{\beta^k + 6}$, $and\ g(\ \beta^{k+1}) = \frac{1}{2} \frac{1}{\left[\left(\beta^{k+1} \right)^2 - 1 \right]}$,

That is for $\beta^0 = -3$,

$when\ k = 0$, we have

$$\beta^1 = \sqrt{\beta^0 + 6}$$

$$\beta^1 = \sqrt{-3 + 6}$$

$$\beta^1 = \sqrt{3}$$

This implies

$$g(\beta^1) - \frac{1}{16} = \frac{1}{2}\frac{1}{[(\beta^1)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{4} - \frac{1}{16}$$

$$= 0.25 - 0.0625$$

$$= 0.1875$$

$when\ k = 1$, we have

$$\beta^2 = \sqrt{\beta^1 + 6}$$

$$\beta^2 = \sqrt{1.7320 + 6}$$

$$\beta^2 = 2.780647$$

This implies

$$g(\beta^2) - \frac{1}{16} = \frac{1}{2}\frac{1}{[(\beta^2)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{13.46399} - \frac{1}{16}$$

$$= 0.0742721 - 0.0625$$

$$= 0.011772$$

Now applying ratio test, we get

$$\frac{g(\beta^2) - \frac{1}{16}}{g(\beta^1) - \frac{1}{16}} = \frac{\frac{1}{2}\frac{1}{[(\beta^2)^2 - 1]} - \frac{1}{16}}{\frac{1}{2}\frac{1}{[(\beta^1)^2 - 1]} - \frac{1}{16}} = \frac{0.011772}{0.1875} = 0.062784 \ < 1$$

$when\ k = 2$, we have

$$\beta^3 = \sqrt{\beta^2 + 6}$$

$$\beta^3 = \sqrt{2.780647 + 6}$$

$$\beta^3 = 2.9632156$$

This implies

$$g(\beta^3) - \frac{1}{16} = \frac{1}{2}\frac{1}{[(\beta^3)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{15.5612933} - \frac{1}{16}$$

$$= 0.0642620 - 0.0625$$

$$= 0.00176200$$

Now applying ratio test, we get

$$\frac{g(\beta^3) - \frac{1}{16}}{g(\beta^2) - \frac{1}{16}} = \frac{\frac{1}{2}\frac{1}{[(\beta^3)^2 - 1]} - \frac{1}{16}}{\frac{1}{2}\frac{1}{[(\beta^2)^2 - 1]} - \frac{1}{16}} = \frac{0.00176200}{0.011772} = 0.1496772001 < 1.$$

$when\ k = 3$, we have

$$\beta^4 = \sqrt{\beta^3 + 6}$$

$$\beta^4 = \sqrt{2.9632156 + 6}$$

$$\beta^4 = 2.9938629$$

This implies

$$g(\beta^4) - \frac{1}{16} = \frac{1}{2}\frac{1}{[(\beta^4)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{15.926430128} - \frac{1}{16}$$

$$= 0.0627887 - 0.0625$$

$$= 0.00028870$$

Now applying ratio test, we get

$$\frac{g(\beta^4) - \frac{1}{16}}{g(\beta^3) - \frac{1}{16}} = \frac{\frac{1}{2}\frac{1}{[(\beta^4)^2 - 1]} - \frac{1}{16}}{\frac{1}{2}\frac{1}{[(\beta^3)^2 - 1]} - \frac{1}{16}} = \frac{0.00028870}{0.00176200} = 0.1638479001 < 1.$$

Use the ratio test at this point. Since $g(\beta^{k+1}) - \frac{1}{16} > 0$ and the sequence $\{\beta^k\}_{k \in N}$ in this case is increases strictly that's how it is

$$\frac{\frac{1}{2}\frac{1}{[(\beta^{k+2})^2 - 1]} - \frac{1}{16}}{\frac{1}{2}\frac{1}{[(\beta^{k+1})^2 - 1]} - \frac{1}{16}} < 1$$

This proved the smoothness of (2.5).

2.3.4 Case 3: $\boldsymbol{\beta^0} \in [-\mathbf{6}, -\mathbf{5}) \cup (\mathbf{3}, +\infty)(\boldsymbol{i.e.}, \boldsymbol{\beta^{k+1}} \in [\mathbf{0}, \mathbf{1}) \cup (\mathbf{3}, +\infty))$.

In this case,

$$\left\|e_{(3)}^{k} - e_3^{\infty}\right\|_{\infty} = 16\left[\frac{1}{16} - g(\beta^{k+1})\right]$$

Consider $$\frac{1}{16} - g(\beta^{k+1}) = \frac{1}{16} - \frac{1}{2}\frac{1}{\left[\left(\beta^{k+1}\right)^2 - 1\right]}$$

For $k = 0$, we have

$$\frac{1}{16} - g(\beta^1) = \frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^1)^2 - 1]}$$

Now choose $\boldsymbol{\beta^0} = -\mathbf{6}, (\boldsymbol{i.e.}, \boldsymbol{\beta^1} = \mathbf{0})$.

$$= \frac{1}{16} - \frac{1}{2}\frac{1}{[(0)^2 - 1]}$$

$$= \frac{1}{16} + \frac{1}{2}$$

$$= 0.0625 + 0.5$$

$$= 0.5625 < +\infty.$$

For $k = 1$, we have

$$\frac{1}{16} - g(\beta^2) = \frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^2)^2 - 1]}$$

$\beta^1 = 0, (i.e., \beta^2 = \sqrt{6} = 2.4495)$

$$= \frac{1}{16} - \frac{1}{2[(\sqrt{6})^2 - 1]}$$

$$= \frac{1}{16} - \frac{1}{10}$$

$$= 0.0625 - 0.1$$

$$= -0.0375 < +\infty.$$

For $k = 2$

$$\frac{1}{16} - g(\beta^3) = \frac{1}{16} - \frac{1}{2[(\beta^3)^2 - 1]}$$

$\beta^2 = 2.4495, (i.e., \beta^3 = 2.90680)$.

$$= \frac{1}{16} - \frac{1}{2[(2.90680)^2 - 1]}$$

$$= \frac{1}{16} - \frac{1}{14.89897}$$

$$= 0.0625 - 0.06712$$

$$= -0.00462 < +\infty.$$

For $k = 3$

$$\frac{1}{16} - g(\beta^4) = \frac{1}{16} - \frac{1}{2[(\beta^4)^2 - 1]}$$

$\beta^3 = 2.90680, (i.e., \beta^4 = 2.98443)$.

$$= \frac{1}{16} - \frac{1}{2[(2.98443)^2 - 1]}$$

$$= \frac{1}{16} - \frac{1}{15.81364}$$

$$= 0.0625 - 0.06324$$

$$= -0.00074 < +\infty.$$

Again for $k = 0$, we have

$$\frac{1}{16} - g(\beta^1) = \frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^1)^2 - 1]}$$

Now choose $\boldsymbol{\beta^0 = \sqrt{10} = 3.1623}$, ($\boldsymbol{i.e., \beta^1 = 3.0269}$).

$$= \frac{1}{2[(3.0269)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{16.3242} - \frac{1}{16}$$

$$= 0.06126 - 0.0625$$

$$= -0.00124 < +\infty.$$

For $k = 1$

$$\frac{1}{16} - g(\beta^2) = \frac{1}{2[(\beta^2)^2 - 1]} - \frac{1}{16}$$

$\beta^1 = 3.0269, (i.e., \beta^2 = 3.0045)$.

$$= \frac{1}{2[(3.0045)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{16.0540} - \frac{1}{16}$$

$$= 0.06229 - 0.0625$$

$$= -0.00021 < +\infty.$$

For $k = 2$

$$\frac{1}{16} - g(\beta^3) = \frac{1}{2[(\beta^3)^2-1]} - \frac{1}{16}$$

$\beta^2 = 3.0045, (i.e., \beta^3 = 3.00075)$.

$$= \frac{1}{2[(3.00075)^2-1]} - \frac{1}{16}$$

$$= \frac{1}{16.00900} - \frac{1}{16}$$

$$= 0.06246 - 0.0625$$

$$= -0.00004 < +\infty.$$

For $k = 3$

$$\frac{1}{16} - g(\beta^4) = \frac{1}{2[(\beta^4)^2-1]} - \frac{1}{16}$$

$\beta^2 = 3.00075, (i.e., \beta^3 = 3.00012)$.

$$= \frac{1}{2[(3.00012)^2-1]} - \frac{1}{16}$$

$$= \frac{1}{16.00144} - \frac{1}{16}$$

$$= 0.06249 - 0.0625$$

$$= -0.00001 < +\infty.$$

Thus

$$\sum_{k=0}^{+\infty}\left(\frac{1}{16} - g(\beta^{k+1})\right) = \sum_{k=0}^{+\infty}\left(\frac{1}{16} - \frac{1}{2}\frac{1}{\left[(\beta^{k+1})^2-1\right]}\right) < +\infty.$$

So, we've got two subcases.

2.3.4.1 Case 3.1 $\beta^0 \in (3, +\infty)(i.e., \beta^{k+1} \in (3, +\infty))$. Since in this case the sequence $\{\beta^k\}_{k\in N}$ is decreases strictly.

For $\beta^0 = 4,\ \beta^{k+1} = \sqrt{\beta^k + 6},\ when\ k = 0$, we have

$$\beta^1 = \sqrt{\beta^0 + 6}$$

$$\beta^1 = \sqrt{4 + 6}$$

$$\beta^1 = \sqrt{10}$$

This implies

$$\frac{1}{16} - g(\beta^1) = \frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^1)^2 - 1]}$$

$$= \frac{1}{16} - \frac{1}{18}$$

$$= 0.0625 - 0.05555$$

$$= 0.00695$$

$when\ k = 1$, we have

$$\beta^2 = \sqrt{\beta^1 + 6}$$

$$\beta^2 = \sqrt{3.16227 + 6}$$

$$\beta^2 = 3.026924$$

This implies

$$\frac{1}{16} - g(\beta^2) = \frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^2)^2 - 1]}$$

$$= \frac{1}{16} - \frac{1}{16.32453}$$

$$= 0.0625 - 0.0612575$$

$$= 0.0012425$$

Now applying ratio test, we get

$$\frac{\frac{1}{16} - g(\beta^2)}{\frac{1}{16} - g(\beta^1)} = \frac{\frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^2)^2 - 1]}}{\frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^1)^2 - 1]}} = \frac{0.0012425}{0.00695} = 0.17878 < 1$$

$when\ k = 2$, we have

$$\beta^3 = \sqrt{\beta^2 + 6}$$

$$\beta^3 = \sqrt{3.026924 + 6}$$

$$\beta^3 = 3.004484$$

This implies

$$\frac{1}{16} - g(\beta^3) = \frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^3)^2 - 1]}$$

$$= \frac{1}{16} - \frac{1}{16.053848}$$

$$= 0.0625 - 0.062290$$

$$= 0.00021$$

Now applying ratio test, we get

$$\frac{\frac{1}{16}-g(\beta^2)}{\frac{1}{16}-g(\beta^1)} = \frac{\frac{1}{16}-\frac{1}{2}\frac{1}{[(\beta^2)^2-1]}}{\frac{1}{16}-\frac{1}{2}\frac{1}{[(\beta^1)^2-1]}} = \frac{0.00021}{0.0012425} = 0.1690141 < 1$$

$when\ k = 3$, we have

$$\beta^4 = \sqrt{\beta^3 + 6}$$

$$\beta^4 = \sqrt{3.004484 + 6}$$

$$\beta^4 = 3.000747$$

This implies

$$\frac{1}{16} - g(\beta^4) = \frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^4)^2-1]}$$

$$= \frac{1}{16} - \frac{1}{16.008965}$$

$$= 0.0625 - 0.06246$$

$$= 0.00004$$

Now applying ratio test, we get

$$\frac{\frac{1}{16} - g(\beta^4)}{\frac{1}{16} - g(\beta^3)} = \frac{\frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^4)^2 - 1]}}{\frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^3)^2 - 1]}} = \frac{0.00004}{0.00021} = 0.190476 < 1$$

Use the ratio test at this point. In this case the sequence $\{\beta^k\}_{k\in N}$ increases strictly to such an extent that

$$\frac{\frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^{k+2})^2 - 1]}}{\frac{1}{16} - \frac{1}{2}\frac{1}{[(\beta^{k+1})^2 - 1]}} < 1$$

Hence the smoothness of (2.5) is therefore proven.

2.3.4.2 Case 3.2

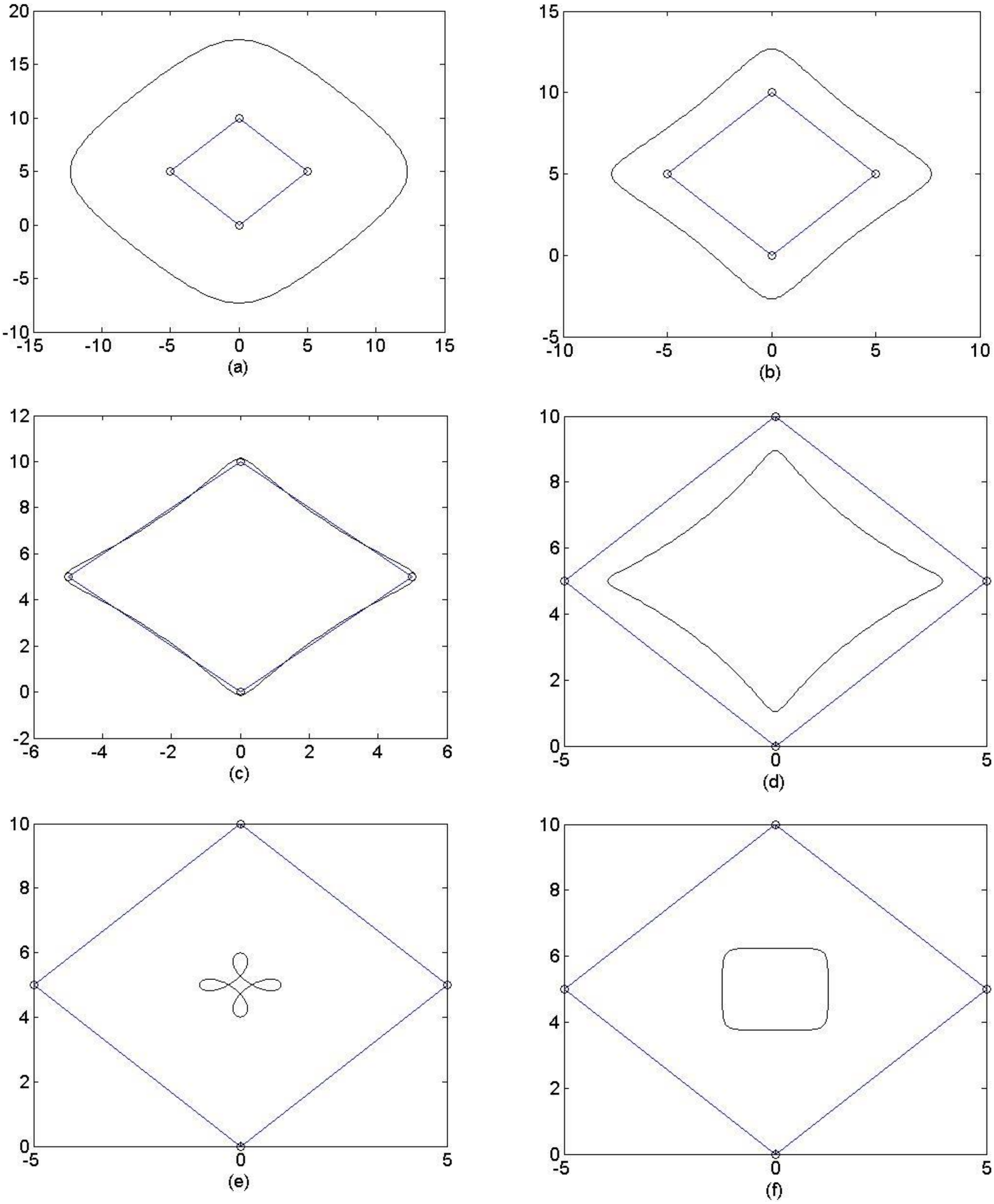
$\beta^0 \in [-6, -5)$. We get $\beta^1 \in [0,1)$ and $\beta^k \in (1,3), k = 2,3,4, \dots$, turning to case 2. Smoothness has therefore been proven.

In combining these three cases, it can be concluded that the non-stationary SS defined by the coefficients in (2.2) is asymptotically equivalent to the stationary scheme in (2.1) with $\beta_1^k = \frac{1}{16}, \beta_2^k = \frac{5}{8}, \beta_3^k = \frac{5}{16}$, and developed continuous limit curve of C^3.

The verification of theorem 1 is done.

2.4 Graphical View

We would like to give an example in this section to show the benefit of the scheme (2.2). As we mentioned in section 2.2, the curves generated tend to approximate the initial polygon of control when $\beta^0 \to +\infty$.
In Figure 2.1, generation of wide range of C^3-continuous limiting curves for different parameter values using the scheme (2.1) (a) $\beta^0 = -5.5$, (b) $\beta^0 = -5.2$, (c) $\beta^0 = -5$, (d) $\beta^0 = -3.8$, (e) $\beta^0 = -2.5$, (f) $\beta^0 = 0$, (g) $\beta^0 = 2$, (h) $\beta^0 = 10$, (i) $\beta^0 = 1000$.


(a)
(b)
(c)
(d)
(e)
(f)

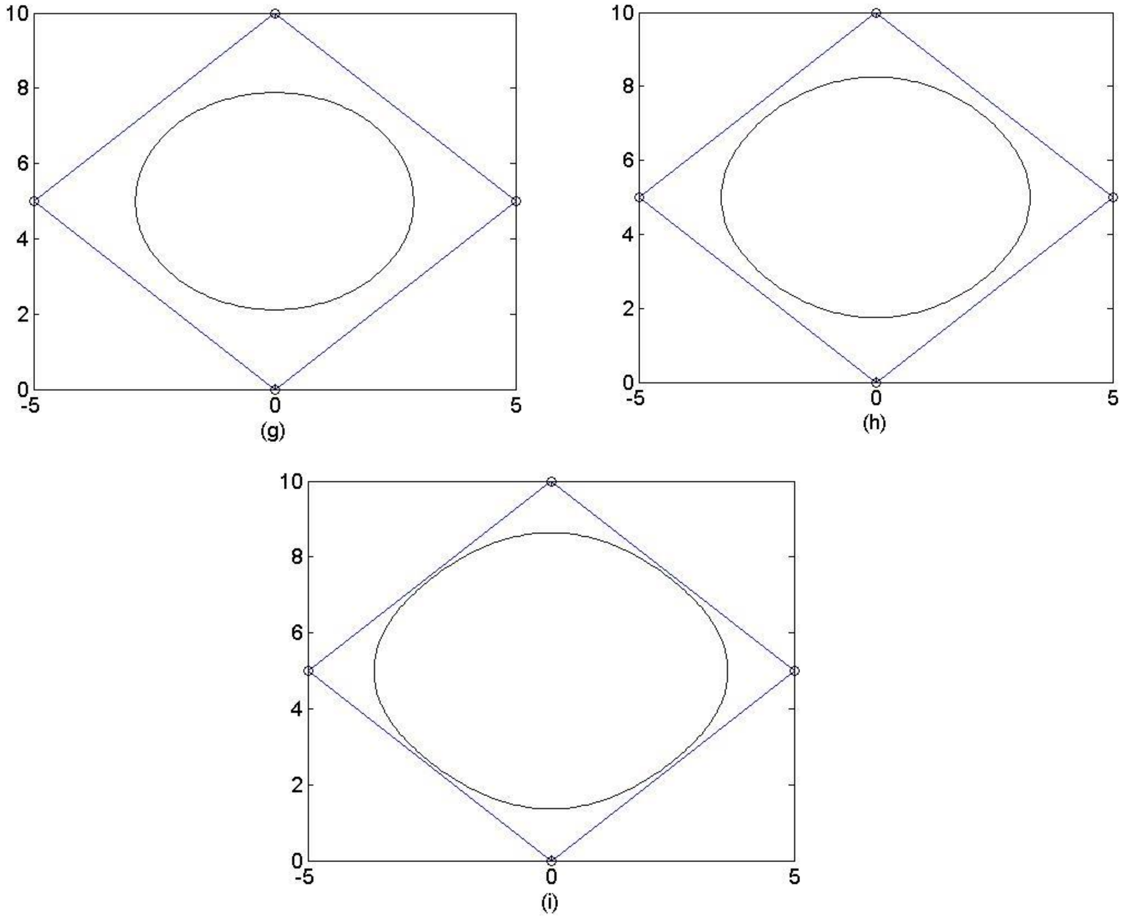


Figure 2.1: Generating wide range of C^3-continuous limiting curves using the scheme (2.2) for different values of parameter. (a) $\beta^0 = -5.5$, (b) $\beta^0 = -5.2$, (c) $\beta^0 = -5$, (d) $\beta^0 = -3.8$, (e) $\beta^0 = -2.5$, (f) $\beta^0 = 0$, (g) $\beta^0 = 2$, (h) $\beta^0 = 10$, (i) $\beta^0 = 1000$.

Chapter # 3

A non-stationary four-point subdivision technique

3.1 Stationary four-point subdivision technique

Kim *et al.* [13] proposed a binary subdivision scheme with four points that generates a smooth C^3-continuous limiting curve. Given the set of control points $q^0 = \{{q_i}^0\}_{i\in Z}$ at level 0, the binary four point SS for the design of curves generates a new set of control points $\{{q_i}^{k+1}\}_{i\in Z}$ at level k+1 using the following subdivision rules

$$q_{2i}^{k+1} = -\beta_0^k q_{i-2}^k + \beta_1^k q_{i-1}^k + \beta_2^k q_i^k + \beta_3^k q_{i+1}^k - \beta_4^k q_{i+2}^k$$

$$q_{2i+1}^{k+1} = -\frac{1}{16} q_{i-1}^k + \frac{9}{16} q_i^k + \frac{9}{16} q_{i+1}^k - \frac{1}{16} q_{i+2}^k. \qquad \ldots\ldots\ldots\ldots(3.1)$$

Where $q^0 = \{q_i^0\}_{i\in Z}$ is the set of initial control point at level 0 and the mask of the scheme, the relationship $\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$ shell be met. The scheme coefficients (3.1) are

$$\beta_0 = -\frac{3}{128}, \beta_1 = \frac{12}{128}, \beta_2 = \frac{110}{128}, \beta_3 = \frac{12}{128}, \beta_4 = -\frac{3}{128}.$$

They found that the scheme is C^1-continuous when and the scheme is C^2-continuous when $-0.12 < \beta < 0.21$ and the scheme is C^3-continuous when $-0.88 < \beta < 0.13$. For the range of $-0.88 < \beta < 0.13$, the proposed scheme is non-stationary scheme.

3.2 Non-stationary four-point subdivision technique

The refining rules of the binary non-stationary SS of four points are defined as

$$q_{2i}^{k+1} = -\beta_0^k q_{i-2}^k + \beta_1^k q_{i-1}^k + \beta_2^k q_i^k + \beta_3^k q_{i+1}^k - \beta_4^k q_{i+2}^k$$

$$q_{2i+1}^{k+1} = -\frac{1}{16} q_{i-1}^k + \frac{9}{16} q_i^k + \frac{9}{16} q_{i+1}^k - \frac{1}{16} q_{i+2}^k. \qquad \ldots\ldots\ldots\ldots(3.2)$$

Where $q^0 = \{q_i^0\}_{i\in Z}$ is the set of initial control point at level 0 and the mask of the scheme, the relationship $\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$ shell be met.

The binary non-stationary four point subdivision scheme (3.2) is counter part of stationary scheme [13]. The mask of the scheme are given by

$$\beta_0^k = -g(\beta^{k+1}) = \beta_4^k, \quad \beta_1^k = 4g(\beta^{k+1}) = \beta_3^k, \quad \beta_2 = 1 - 6g(\beta^{k+1});$$

$$g(\beta^{k+1}) = \frac{1}{2}\left[\frac{(\beta^{k+1})^2 - 1}{(\beta^{k+1})^2 + 60}\right] \qquad \ldots\ldots\ldots(3.3)$$

With

$$\beta^{k+1} = \sqrt{\beta^k + 2}, \qquad and \qquad \beta^0 \in [-2, +\infty). \qquad \ldots\ldots\ldots\ldots(3.4)$$

In this way, the coefficients β_i^k at each different steps k can be calculated using the given formula and given an initial parameter $\beta^0 \in [-2, +\infty)$.

Starting with any $\beta^0 \geq -2$, we always have $\beta^k + 2 \geq 0, \forall k \in Z_{+,}$ so β^{k+1} is always well-established. A vast range of definitions enables to achieve significant variations in the form of the smooth curves.

3.2.1 Remark 1. As the initial values of β^0 increases in its definition range, the behaviour of the smooth curve changes significantly from the figure (3.1) and tends to approximate the initial control polygon as $\beta^0 \to +\infty$.

3.2.2 Remark 2. From (3.3), if we have $\beta^0 = 2$ then $\beta^k = 2$, $\forall\ k \in Z_+$ and the sequence $\{\beta^k\}_{k\in N}$ is stationary. So that the non-stationary SS is then retrograde to the stationary SS.

That's k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 2}$$

$$\beta^1 = \sqrt{2 + 2}$$

$$\beta^1 = \sqrt{4}$$

$$\beta^1 = 2$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 2}$$

$$\beta^2 = \sqrt{2 + 2}$$

$$\beta^2 = \sqrt{4}$$

$$\beta^2 = 2$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 2}$$

$$\beta^3 = \sqrt{2 + 2}$$

$$\beta^3 = \sqrt{4}$$

$$\beta^3 = 2$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 2}$$

$$\beta^4 = \sqrt{2 + 2}$$

$$\beta^4 = 2$$

If $\beta^0 > 2$, then $\beta^k > 2$, the sequence $\{\beta^k\}_{k\in N}$ is strictly reduced that β^k converges to 2 as the $k \to \infty$.

Now choose $\boldsymbol{\beta^0 = 3}$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 2}$$

$$\beta^1 = \sqrt{3 + 2}$$

$$\beta^1 = \sqrt{5}$$

$$\beta^1 = 2.23607$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 2}$$

$$\beta^2 = \sqrt{2.23607 + 2}$$

$$\beta^2 = \sqrt{4.23607}$$

$$\beta^2 = 2.05817$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 2}$$

$$\beta^3 = \sqrt{2.05817 + 2}$$

$$\beta^3 = \sqrt{4.05817}$$

$$\beta^3 = 2.014490$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 2}$$

$$\beta^4 = \sqrt{2.014490 + 2}$$

$$\beta^4 = \sqrt{4.01449}$$

$$\beta^4 = 2.00361$$

Now choose $\boldsymbol{\beta^0 = 4}$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 2}$$

$$\beta^1 = \sqrt{4 + 2}$$

$$\beta^1 = \sqrt{6}$$

$$\beta^1 = 2.44949$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 2}$$

$$\beta^2 = \sqrt{2.44949 + 2}$$

$$\beta^2 = \sqrt{4.44949}$$

$$\beta^2 = 2.10938$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 2}$$

$$\beta^3 = \sqrt{2.10938 + 2}$$

$$\beta^3 = \sqrt{4.10938}$$

$$\beta^3 = 2.02716$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 2}$$

$$\beta^4 = \sqrt{2.02716 + 2}$$

$$\beta^4 = \sqrt{4.02716}$$

$$\beta^4 = 2.00678$$

Now choose $\boldsymbol{\beta^0 = 5}$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 2}$$

$$\beta^1 = \sqrt{5 + 2}$$

$$\beta^1 = \sqrt{7}$$

$$\beta^1 = 2.64575$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 2}$$

$$\beta^2 = \sqrt{2.64575 + 2}$$

$$\beta^2 = \sqrt{4.64575}$$

$$\beta^2 = 2.155400$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 2}$$

$$\beta^3 = \sqrt{2.155400 + 2}$$

$$\beta^3 = \sqrt{4.155400}$$

$$\beta^3 = 2.03848$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 2}$$

$$\beta^4 = \sqrt{2.03848 + 2}$$

$$\beta^4 = \sqrt{4.03848}$$

$$\beta^4 = 2.009597$$

Now choose $\boldsymbol{\beta^0 = 6}$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 2}$$

$$\beta^1 = \sqrt{6 + 2}$$

$$\beta^1 = \sqrt{8}$$

$$\beta^1 = 2.828427$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 2}$$

$$\beta^2 = \sqrt{2.828427 + 2}$$

$$\beta^2 = \sqrt{4.828427}$$

$$\beta^2 = 2.19737$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 2}$$

$$\beta^3 = \sqrt{2.19737 + 2}$$

$$\beta^3 = \sqrt{4.19737}$$

$$\beta^3 = 2.048748$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 2}$$

$$\beta^4 = \sqrt{2.048748 + 2}$$

$$\beta^4 = \sqrt{4.048748}$$

$$\beta^4 = 2.012150$$

Similarly we can check for β^0 = 7 and so on.

If $\beta^0 < 2$, then $\beta^k < 2$, the sequence $\{\beta^k\}_{k\in N}$ is increases strictly and β^k converges to 2 as the $k \to \infty$.

Choose $\boldsymbol{\beta^0 = 1}$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 2}$$

$$\beta^1 = \sqrt{1 + 2}$$

$$\beta^1 = \sqrt{3}$$

$$\beta^1 = 1.732050$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 2}$$

$$\beta^2 = \sqrt{1.732050 + 2}$$

$$\beta^2 = \sqrt{3.732050}$$

$$\beta^2 = 1.931851$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 2}$$

$$\beta^3 = \sqrt{1.931851 + 2}$$

$$\beta^3 = \sqrt{3.931851}$$

$$\beta^3 = 1.982889$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 2}$$

$$\beta^4 = \sqrt{1.982889 + 2}$$

$$\beta^4 = \sqrt{3.982889}$$

$$\beta^4 = 1.9957177$$

Choose $\boldsymbol{\beta^0 = 0}$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 2}$$

$$\beta^1 = \sqrt{0 + 2}$$

$$\beta^1 = \sqrt{2}$$

$$\beta^1 = 1.414214$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 2}$$

$$\beta^2 = \sqrt{1.414214 + 2}$$

$$\beta^2 = \sqrt{3.414214}$$

$$\beta^2 = 1.8477592$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 2}$$

$$\beta^3 = \sqrt{1.8477592 + 2}$$

$$\beta^3 = \sqrt{3.8477592}$$

$$\beta^3 = 1.961571$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 2}$$

$$\beta^4 = \sqrt{1.961571 + 2}$$

$$\beta^4 = \sqrt{3.961571}$$

$$\beta^4 = 1.990369$$

Choose $\boldsymbol{\beta^0 = -1}$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 2}$$

$$\beta^1 = \sqrt{-1 + 2}$$

$$\beta^1 = \sqrt{1}$$

$$\beta^1 = 1.$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 2}$$

$$\beta^2 = \sqrt{1 + 2}$$

$$\beta^2 = 1.732051$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 2}$$

$$\beta^3 = \sqrt{1.732051 + 2}$$

$$\beta^3 = \sqrt{3.732051}$$

$$\beta^3 = 1.931852$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 2}$$

$$\beta^4 = \sqrt{1.931852 + 2}$$

$$\beta^4 = \sqrt{3.931852}$$

$$\beta^4 = 1.98289$$

Choose $\boldsymbol{\beta^0 = -2}$.

Put k = 0, we get

$$\beta^1 = \sqrt{\beta^0 + 2}$$

$$\beta^1 = \sqrt{-2 + 2}$$

$$\beta^1 = 0.$$

Put k = 1, we get

$$\beta^2 = \sqrt{\beta^1 + 2}$$

$$\beta^2 = \sqrt{0 + 2}$$

$$\beta^2 = \sqrt{2}$$

$$\beta^2 = 1.414214$$

Put k = 2, we get

$$\beta^3 = \sqrt{\beta^2 + 2}$$

$$\beta^3 = \sqrt{1.414214 + 2}$$

$$\beta^3 = \sqrt{3.414214}$$

$$\beta^3 = 1.8477592$$

Put k = 3, we get

$$\beta^4 = \sqrt{\beta^3 + 2}$$

$$\beta^4 = \sqrt{1.8477592 + 2}$$

$$\beta^4 = \sqrt{3.8477592}$$

$$\beta^4 = 1.96157$$

Therefore, in the definition domain of any β^0 we always have

$$\lim_{k\to+\infty} \beta^k = 2.$$

3.3 Smoothness Analysis

In this section, we will illustrate that, given the initial discrete polygon, the subdivision technique developed in section 3.2 gives a smooth C^3 continuous curve for any selection of the initial parameter values β^0 in its definition range. To show this fact, we quote Dyn and Levin's well-known results [25], which relate the smoothness of a non-stationary technique with its asymptotically equivalent stationary technique counterpart.

3.3.1 Theorem. A non-stationary SS defined by the masks in equation (3.3) is asymptotically equivalent to the stationary system with masks in equation (3.1), C^3-continuous limit curves are therefore generated.

Proof. In order to check that the proposed non-stationary technique converges to C^3-continuous smooth curve, its second divided difference mask should be obtained. The scheme mask is

$$m^k = \left[-g(\beta^{k+1}), -\frac{1}{16}, 4g(\beta^{k+1}), \frac{9}{16}, 1-6g(\beta^{k+1}), \frac{9}{16}, 4g(\beta^{k+1}), -\frac{1}{16}, -g(\beta^{k+1})\right]$$

Then it turns out its first divided difference masks are

$$e_{(1)}^k = 2\left[-\beta, \left(\beta-\frac{1}{16}\right), \left(3\beta+\frac{1}{16}\right), \left(\frac{1}{2}-3\beta\right), \left(\frac{1}{2}-3\beta\right), \left(3\beta+\frac{1}{16}\right), \left(\beta-\frac{1}{16}\right), -\beta\right]$$

Then it turns out its 2nd divided difference masks are

$$e_{(2)}^k = 4\left[-\beta, \left(2\beta-\frac{1}{16}\right), \left(\beta+\frac{1}{8}\right), \left(\frac{3}{8}-4\beta\right), \left(\beta+\frac{1}{8}\right), \left(2\beta-\frac{1}{16}\right), -\beta\right]$$

Then it turns out its 3rd divided difference masks are

$$e_{(3)}^k = 8\left[-\beta, \left(3\beta-\frac{1}{16}\right), \left(-2\beta+\frac{3}{16}\right), \left(-2\beta+\frac{3}{16}\right), \left(3\beta-\frac{1}{16}\right), -\beta\right]$$

The application of Remark 2 now provides

$$e_{(3)}^{\infty} = \lim_{k\to+\infty} e_3^k = 8\left[-\frac{3}{128}, \frac{1}{128}, \frac{18}{128}, \frac{18}{128}, \frac{1}{128}, -\frac{3}{128}\right]$$

This is just the coefficients of the third divided differences of the stationary technique with coefficients in equation (3.1). In this case, the stationary technique is C^3-continuous, the technique associated with e_3^{∞} will be C^3 smooth. If it is as

$$\sum_{k=0}^{+\infty}\left\|e_{(3)}^k - e_3^{\infty}\right\|_{\infty} < +\infty. \qquad \text{………(3.5)}$$

The two techniques are then equivalent asymptotically. And one can conclude this that the e_3^{∞} of the technique is C^3, since then

$$e_{(3)}^k - e_3^{\infty} = 8[-2g(\beta^{k+1}) + \frac{6}{128}, -3\,(-2g(\beta^{k+1}) + \frac{6}{128}, 2(-2g(\beta^{k+1}) + \frac{6}{128})]$$

$$\left\|e_{(3)}^k - e_3^{\infty}\right\|_{\infty} = 8max\left\{3\left|-2g(\beta^{k+1}) + \frac{6}{128}\right|, |-3|\left|-2g(\beta^{k+1}) + \frac{6}{128}\right|, 2\left|-2g(\beta^{k+1}) + \frac{6}{128}\right|\right\}$$

$$= 24\left|-2g(\beta^{k+1}) + \frac{6}{128}\right|$$

$$= 48\left|\frac{3}{128} - g(\beta^{k+1})\right|$$

Now we are proving the series smoothness

$$\sum_{k=0}^{+\infty}\left|\frac{3}{128} - g(\beta^{k+1})\right|. \qquad \text{………..(3.6)}$$

Which depends on the $g(\beta^{k+1})$ function. Now, since $g(\beta^{k+1})$ is expressed in terms of the β^{k+1}, the behaviour of β^{k+1} varies in the interval [0, +∞). From now on

$\boldsymbol{\frac{3}{128} - g(\beta^{k+1}) = 0 \Leftrightarrow \beta^{k+1} = 2}$**(i.e.,** $\boldsymbol{\beta^{k} = 2}$**))**.

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}\right]$

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(2)^2-1}{(2)^2+60}\right]$

$= \frac{3}{128} - \frac{3}{128}$

= 0.

$\boldsymbol{\frac{3}{128} - g(\beta^{k+1}) > 0 \Leftrightarrow \beta^{k+1} \in (-1, 2)\left(i.e., \beta^{0} \in [-2, 2)\right)}$.

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}\right]$

For $k = 0$

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^{1})^2-1}{(\beta^{1})^2+60}\right]$

Choose $\boldsymbol{\beta^{0} = -2, (i.e., \beta^{1} = 0)}$.

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(0)^2-1}{(0)^2+60}\right]$

$= \frac{3}{128} + \frac{1}{120}$

$= 0.0234375 + 0.008333$

$= 0.0317705 > 0.$

For $k = 1$

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^{2})^2-1}{(\beta^{2})^2+60}\right]$

Choose $\beta^{1} = 0, \left(i.e., \beta^{2} = \sqrt{2}\right)$.

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\sqrt{2})^2-1}{(\sqrt{2})^2+60}\right]$

$= \frac{3}{128} - \frac{1}{124}$

$= 0.0234375 - 0.008065$

= 0.0153725 > 0.

For $k = 2$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}]$

Choose $\beta^2 = 1.4142, (i.e., \beta^3 = 1.84776)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(1.84776)^2-1}{(1.84776)^2+60}]$

$= \frac{3}{128} - \frac{2.414217}{126.8284}$

$= 0.0234375 - 0.019035$

= 0.0044025 > 0.

For $k = 3$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^4)^2-1}{(\beta^4)^2+60}]$

Choose $\beta^3 = 1.84776, (i.e., \beta^4 = 1.961571)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(1.961571)^2-1}{(1.961571)^2+60}]$

$= \frac{3}{128} - \frac{2.847761}{127.69552}$

$= 0.0234375 - 0.022301$

$= 0.0011365 > 0.$

Again for $k = 0$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^1)^2-1}{(\beta^1)^2+60}]$

Now choose $\boldsymbol{\beta^0 = -1}, (\boldsymbol{i.e., \beta^1 = 1})$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(1)^2-1}{(1)^2+60}]$

$= \frac{3}{128} - 0$

$= 0.0234375 > 0.$

For $k = 1$

$$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^2)^2 - 1}{(\beta^2)^2 + 60}\right]$$

Choose $\beta^1 = 1, \left(i.e., \beta^2 = \sqrt{3} = 1.73205\right).$

$$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\sqrt{3})^2 - 1}{(\sqrt{3})^2 + 60}\right]$$

$$= \frac{3}{128} - \frac{2}{126}$$

$$= 0.0234375 - 0.015873$$

$$= 0.0075645 > 0.$$

For $k = 2$

$$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^3)^2 - 1}{(\beta^3)^2 + 60}\right]$$

Choose $\beta^2 = 1.73205, (i.e., \beta^3 = 1.931851).$

$$= \frac{3}{128} - \frac{1}{2}\left[\frac{(1.931851)^2 - 1}{(1.931851)^2 + 60}\right]$$

$$= \frac{3}{128} - \frac{2.73205}{127.4641}$$

$$= 0.0234375 - 0.021434$$

$$= 0.0020035 > 0.$$

For $k = 3$

$$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^4)^2 - 1}{(\beta^4)^2 + 60}\right]$$

Choose $\beta^3 = 1.931851, (i.e., \beta^4 = 1.98289).$

$$= \frac{3}{128} - \frac{1}{2}\left[\frac{(1.98289)^2 - 1}{(1.98289)^2 + 60}\right]$$

$$= \frac{3}{128} - \frac{2.93185}{127.8637}$$

$$= 0.0234375 - 0.02293$$

$$= 0.0005075 > 0.$$

Again for $k = 0$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^1)^2-1}{(\beta^1)^2+60}]$

Now choose $\boldsymbol{\beta^0 = 0}$, $\boldsymbol{(i.e., \beta^1 = \sqrt{2})}$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(\sqrt{2})^2-1}{(\sqrt{2})^2+60}]$

$= \frac{3}{128} - \frac{1}{124}$

$= 0.0234375 - 0.008065$

$= 0.0153725 > 0.$

For $k = 1$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^2)^2-1}{(\beta^2)^2+60}]$

$\beta^1 = 1.4142, (i.e., \beta^2 = 1.84776).$

$= \frac{3}{128} - \frac{1}{2}[\frac{(1.84776)^2-1}{(1.84776)^2+60}]$

$= \frac{3}{128} - \frac{2.414217}{126.8284}$

$= 0.0234375 - 0.019035$

= 0.0044025 > 0.

For $k = 2$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}]$

$\beta^2 = 1.84776, (i.e., \beta^3 = 1.961571).$

$= \frac{3}{128} - \frac{1}{2}[\frac{(1.961571)^2-1}{(1.961571)^2+60}]$

$= \frac{3}{128} - \frac{2.847761}{127.69552}$

$= 0.0234375 - 0.022301$

$= 0.0011365 > 0.$

For $k = 3$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^4)^2-1}{(\beta^4)^2+60}]$

Choose $\beta^3 = 1.96157, (i.e., \beta^4 = 1.99037)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(1.99037)^2-1}{(1.99037)^2+60}]$

$= \frac{3}{128} - \frac{2.96157}{127.9231}$

$= 0.0234375 - 0.023151$

$= 0.0002865 > 0.$

$\boldsymbol{\frac{3}{128} - g(\beta^{k+1}) < 0 \Leftrightarrow \beta^{k+1} \in (2, +\infty)}$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}]$

For $k = 0$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^1)^2-1}{(\beta^1)^2+60}]$

Choose $\boldsymbol{\beta^0 = 3, (i.e., \beta^1 = \sqrt{5})}$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(\sqrt{5})^2-1}{(\sqrt{5})^2+60}]$

$= \frac{3}{128} - \frac{4}{130}$

$= 0.0234375 - 0.03076923$

$= -0.00733173 < 0.$

For $k = 1$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^2)^2-1}{(\beta^2)^2+60}]$

Choose $\beta^1 = 2.236, (i.e., \beta^2 = 2.0582)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(2.0582)^2-1}{(2.0582)^2+60}]$

$= \frac{3}{128} - \frac{3.23619}{128.4724}$

$= 0.0234375 - 0.0251898$

$= -0.0017523 < 0.$

For $k = 2$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}]$

Choose $\beta^2 = 2.0582, (i.e., \beta^3 = 2.014497)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(2.014497)^2-1}{(2.014497)^2+60}]$

$= \frac{3}{128} - \frac{3.058198}{128.116396}$

$= 0.0234375 - 0.023870$

$= -0.0004325 < 0.$

For $k = 3$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^4)^2-1}{(\beta^4)^2+60}]$

Choose $\beta^3 = 2.014497, (i.e., \beta^4 = 2.003621)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(2.003621)^2-1}{(2.003621)^2+60}]$

$= \frac{3}{128} - \frac{3.014497}{128.02899}$

$= 0.0234375 - 0.0235454$

$= -0.0001079 < 0.$

Again for $k = 0$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^1)^2-1}{(\beta^1)^2+60}]$

Now choose $\boldsymbol{\beta^0 = 4, \left(i.e., \beta^1 = \sqrt{6}\right)}$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(\sqrt{6})^2-1}{(\sqrt{6})^2+60}]$

$= \frac{3}{128} - \frac{5}{132}$

$= 0.0234375 - 0.037879$

$= -0.0144415 < 0.$

For $k = 1$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^2)^2-1}{(\beta^2)^2+60}]$

Choose $\beta^1 = \sqrt{6} = 2.449, (i.e., \beta^2 = 2.10927)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(2.10927)^2-1}{(2.10927)^2+60}]$

$= \frac{3}{128} - \frac{3.44902}{128.89804}$

$= 0.0234375 - 0.026758$

$= -0.0033205 < 0.$

For $k = 2$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}]$

Choose $\beta^2 = 2.10927, (i.e., \beta^3 = 2.027133)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(2.027133)^2-1}{(2.027133)^2+60}]$

$= \frac{3}{128} - \frac{3.10927}{128.21854}$

$= 0.0234375 - 0.0242498$

$= -0.0008123 < 0.$

For $k = 3$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^4)^2-1}{(\beta^4)^2+60}]$

Choose $\beta^3 = 2.027133, (i.e., \beta^4 = 2.00677)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(2.00677)^2-1}{(2.00677)^2+60}]$

$= \frac{3}{128} - \frac{3.027126}{128.05425}$

$= 0.0234375 - 0.023639$

$= -0.0002015 < 0.$

Again for $k = 0$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^1)^2-1}{(\beta^1)^2+60}]$

Now choose $\boldsymbol{\beta^0 = 7}, (\boldsymbol{i.e.}, \boldsymbol{\beta^1 = 3})$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(3)^2-1}{(3)^2+60}]$

$= \frac{3}{128} - \frac{8}{138}$

$= 0.0234375 - 0.0579710$

$= -0.0345335 < 0$

For $k = 1$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^2)^2-1}{(\beta^2)^2+60}]$

$\beta^1 = 3, \left(i.e., \beta^2 = \sqrt{5}\right)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(\sqrt{5})^2-1}{(\sqrt{5})^2+60}]$

$= \frac{3}{128} - \frac{4}{130}$

$= 0.0234375 - 0.03076923$

$= -0.00733173 < 0.$

For $k = 2$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}]$

Choose $\beta^2 = 2.236, (i.e., \beta^3 = 2.0582)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(2.0582)^2-1}{(2.0582)^2+60}]$

$= \frac{3}{128} - \frac{3.23619}{128.4724}$

$= 0.0234375 - 0.0251898$

$= -0.0017523 < 0.$

For $k = 3$

$= \frac{3}{128} - \frac{1}{2}[\frac{(\beta^4)^2-1}{(\beta^4)^2+60}]$

Choose $\beta^3 = 2.0582, (i.e., \beta^4 = 2.014497)$.

$= \frac{3}{128} - \frac{1}{2}[\frac{(2.014497)^2-1}{(2.014497)^2+60}]$

$= \frac{3}{128} - \frac{3.058198}{128.116396}$

$= 0.0234375 - 0.023870$

$= -0.0004325 < 0.$

Now, we are talking about smoothness of (3.6) in the three cases.

3.3.2 Case 1: $\boldsymbol{\beta^0 = 2}$ $(\boldsymbol{i.e., \beta^{k+1} = 2})$

Then $\left\|e_{(3)}^k - e_3^\infty\right\|_\infty = 48\left[\frac{3}{128} - \frac{1}{2}[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}]\right]$

For $k = 0$,

$$= 48\left[\frac{3}{128} - \frac{1}{2}[\frac{(\beta^1)^2-1}{(\beta^1)^2+60}]\right]$$

$$= 48\left[\frac{3}{128} - \frac{1}{2}[\frac{(2)^2-1}{(2)^2+60}]\right]$$

$$= 48\left[\frac{3}{128} - \frac{3}{128}\right]$$

$$= 0$$

Smoothness of (3.6) follows.

3.3.3 Case 2: $\boldsymbol{\beta^0 \in [-2, 2)}$$(\boldsymbol{i.e., \beta^{k+1} \in [0, 2)})$.

In this case

$\left\|e_{(3)}^k - e_3^\infty\right\|_\infty = 48\left[\frac{3}{128} - \frac{1}{2}[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}]\right]$

For $k = 0$

$=\frac{3}{128} - \frac{1}{2}[\frac{(\beta^1)^2-1}{(\beta^1)^2+60}]$

Choose $\boldsymbol{\beta^0 = -2}, (\boldsymbol{i.e.}, \boldsymbol{\beta^1 = 0})$).

$= \left[\frac{3}{128} - \frac{1}{2}\left[\frac{(0)^2-1}{(0)^2+60}\right]\right]$

$= \left[\frac{3}{128} + \frac{1}{120}\right]$

= [$0.0234375 + 0.008333$]

= 0.0317705 < $+\infty$.

For $k = 1$

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^2)^2-1}{(\beta^2)^2+60}\right]$

Now choose $\beta^1 = 0, \left(i.e., \beta^2 = \sqrt{2}\right)$.

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\sqrt{2})^2-1}{(\sqrt{2})^2+60}\right]$

$= \frac{3}{128} - \frac{1}{124}$

$= 0.0234375 - 0.008065$

$= 0.0153725 < +\infty.$

For $k = 2$

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}\right]$

$\beta^2 = 1.4142, (i.e., \beta^3 = 1.84776).$

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(1.84776)^2-1}{(1.84776)^2+60}\right]$

$= \frac{3}{128} - \frac{2.414217}{126.8284}$

$= 0.0234375 - 0.019035$

$= 0.0044025 < +\infty.$

For $k = 3$

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^4)^2-1}{(\beta^4)^2+60}\right]$

$\beta^3 = 1.84776, (i.e., \beta^4 = 1.961571).$

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(1.961571)^2-1}{(1.961571)^2+60}\right]$

$$=\frac{3}{128}-\frac{2.847761}{127.69552}$$

$$=0.0234375-0.022301$$

$$=0.0011365<+\infty.$$

Again for $k=0$

$$=\frac{3}{128}-\frac{1}{2}\left[\frac{(\beta^1)^2-1}{(\beta^1)^2+60}\right]$$

Now choose $\boldsymbol{\beta^0=-1,(i.e.,\beta^1=1)}$.

$$=\frac{3}{128}-\frac{1}{2}\left[\frac{(1)^2-1}{(1)^2+60}\right]$$

$$=\frac{3}{128}-0$$

$$=0.0234375<+\infty.$$

For $k=1$

$$=\frac{3}{128}-\frac{1}{2}\left[\frac{(\beta^2)^2-1}{(\beta^2)^2+60}\right]$$

Choose $\beta^1=1,\left(i.e.,\beta^2=\sqrt{3}=1.73205\right)$.

$$=\frac{3}{128}-\frac{1}{2}\left[\frac{(\sqrt{3})^2-1}{(\sqrt{3})^2+60}\right]$$

$$=\frac{3}{128}-\frac{2}{126}$$

$$=0.0234375-0.015873$$

$$=0.0075645<+\infty.$$

For $k=2$

$$=\frac{3}{128}-\frac{1}{2}\left[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}\right]$$

Choose $\beta^2=1.73205,(i.e.,\beta^3=1.931851)$.

$$=\frac{3}{128}-\frac{1}{2}\left[\frac{(1.931851)^2-1}{(1.931851)^2+60}\right]$$

$$=\frac{3}{128}-\frac{2.73205}{127.4641}$$

$$=0.0234375-0.021434$$

$$=0.0020035<+\infty.$$

For $k = 3$

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^4)^2-1}{(\beta^4)^2+60}\right]$

Choose $\beta^3 = 1.931851, (i.e., \beta^4 = 1.98289)$.

$= \frac{3}{128} - \frac{1}{2}\left[\frac{(1.98289)^2-1}{(1.98289)^2+60}\right]$

$= \frac{3}{128} - \frac{2.93185}{127.8637}$

$= 0.0234375 - 0.02293$

$= 0.0005075 < +\infty.$

Thus

$$\sum_{k=0}^{+\infty}\left(\frac{3}{128} - g(\beta^{k+1})\right) = \sum_{k=0}^{+\infty}\left(\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}\right]\right) < +\infty.$$

Now for $\beta^0 = -2,\ \beta^{k+1} = \sqrt{\beta^k + 2},\ when\ k = 0$, we have

$$\beta^1 = \sqrt{\beta^0 + 2}$$

$$\beta^1 = \sqrt{-2 + 2}$$

$$\beta^1 = 0$$

This implies

$$\frac{3}{128} - g(\beta^1) = \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^1)^2 - 1}{(\beta^1)^2 + 60}\right]$$

$$= \frac{3}{128} + \frac{1}{120}$$

$$= 0.0234375 + 0.008333$$

$$= 0.0317705$$

$when\ k = 1$, we have

$$\beta^2 = \sqrt{\beta^1 + 2}$$

$$\beta^2 = \sqrt{0 + 2}$$

$$\beta^2 = \sqrt{2}$$

This implies

$$\frac{3}{128} - g(\beta^2) = \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^2)^2 - 1}{(\beta^2)^2 + 60}\right]$$

$$= \frac{3}{128} - \frac{1}{124}$$

$$= 0.0234375 - 0.008064516$$

$$= 0.015373$$

Now applying ratio test, we get

$$\frac{\frac{3}{128} - g(\beta^2)}{\frac{3}{128} - g(\beta^1)} = \frac{\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^2)^2 - 1}{(\beta^2)^2 + 60}\right]}{\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^1)^2 - 1}{(\beta^1)^2 + 60}\right]} = \frac{0.015373}{0.0317705} = 0.483877 \quad < 1$$

$when\ k = 2$, we have

$$\beta^3 = \sqrt{\beta^2 + 2}$$

$$\beta^3 = \sqrt{1.4142 + 2}$$

$$\beta^3 = 1.847755$$

This implies

$$\frac{3}{128} - g(\beta^3) = \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^3)^2 - 1}{(\beta^3)^2 + 60}\right]$$

$$= \frac{3}{128} - \frac{2.4142}{126.8284}$$

$$= 0.0234375 - 0.0190352$$

$$= 0.0044023$$

Now applying ratio test, we get

$$\frac{\frac{3}{128} - g(\beta^3)}{\frac{3}{128} - g(\beta^2)} = \frac{\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^3)^2 - 1}{(\beta^3)^2 + 60}\right]}{\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^2)^2 - 1}{(\beta^2)^2 + 60}\right]} = \frac{0.0044023}{0.015373} = 0.286366 \quad < 1.$$

$when\ k = 3$, we have

$$\beta^4 = \sqrt{\beta^3 + 2}$$

$$\beta^4 = \sqrt{1.847755 + 2}$$

$$\beta^4 = 1.961569$$

This implies

$$\frac{3}{128} - g(\beta^4) = \frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^4)^2 - 1}{(\beta^4)^2 + 60}\right]$$

$$= \frac{3}{128} - \frac{2.847752}{127.69550}$$

$$= 0.0234375 - 0.0223011$$

$$= 0.0011364$$

Now applying ratio test, we get

$$\frac{\frac{3}{128} - g(\beta^4)}{\frac{3}{128} - g(\beta^3)} = \frac{\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^4)^2 - 1}{(\beta^4)^2 + 60}\right]}{\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^3)^2 - 1}{(\beta^3)^2 + 60}\right]} = \frac{0.0011364}{0.0044023} = 0.2581378 < 1.$$

We use the ratio test to this end. Since $\frac{3}{128} - g(\beta^{k+1}) > 0$ and the sequence $\{\beta^k\}_{k \in N}$ is strictly increases in this case, we got it

$$\frac{\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^{k+2})^2 - 1}{(\beta^{k+2})^2 + 60}\right]}{\frac{3}{128} - \frac{1}{2}\left[\frac{(\beta^{k+1})^2 - 1}{(\beta^{k+1})^2 + 60}\right]} < 1.$$

The smoothness of (3.6) has therefore been proven.

3.3.4 Case 3: $\boldsymbol{\beta^0} \in (\mathbf{2}, +\infty)(\boldsymbol{i.e}\ \boldsymbol{\beta^{k+1}} \in (\mathbf{2}, +\infty)$.

In this case

$$\left\| e_{(3)}^k - e_3^\infty \right\|_\infty = 48\left[\frac{1}{2}\left\{\frac{(\beta^{k+1})^2 - 1}{(\beta^{k+1})^2 + 60}\right\} - \frac{3}{128}\right]$$

For $k = 0$

$$= \frac{1}{2}\left[\frac{(\beta^1)^2 - 1}{(\beta^1)^2 + 60}\right] - \frac{3}{128}$$

Choose $\boldsymbol{\beta^0 = 3, \left(i.e., \beta^1 = \sqrt{5}\right)}$).

$$= \left[\frac{1}{2}\left\{\frac{(\sqrt{5})^2-1}{\left(\sqrt{5}\right)^2+60}\right\} - \frac{3}{128}\right]$$

$$= \left[\frac{4}{130} - \frac{3}{128}\right]$$

$$= [0.03077 - 0.0234375]$$

$$= 0.0073325 < +\infty.$$

For $k = 1$

$$= \frac{1}{2}\left[\frac{(\beta^2)^2-1}{(\beta^2)^2+60}\right] - \frac{3}{128}$$

Choose $\beta^1 = 2.236, (i.e., \beta^2 = 2.0582)$.

$$= \frac{1}{2}\left[\frac{(2.0582)^2-1}{(2.0582)^2+60}\right] - \frac{3}{128}$$

$$= \frac{3.23619}{128.4724} - \frac{3}{128}$$

$$= 0.0251898 - 0.0234375$$

$$= 0.0017523 < +\infty.$$

For $k = 2$

$$= \frac{1}{2}\left[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}\right] - \frac{3}{128}$$

Choose $\beta^2 = 2.0582, (i.e., \beta^3 = 2.014497)$.

$$= \frac{1}{2}\left[\frac{(2.014497)^2-1}{(2.014497)^2+60}\right] - \frac{3}{128}$$

$$= \frac{3.058198}{128.116396} - \frac{3}{128}$$

$$= 0.023870 - 0.0234375$$

$$= 0.0004325 < +\infty.$$

For $k = 3$

$$= \frac{1}{2}\left[\frac{(\beta^4)^2-1}{(\beta^4)^2+60}\right] - \frac{3}{128}$$

Choose $\beta^3 = 2.014497, (i.e., \beta^4 = 2.003621)$.

$= \frac{1}{2}\left[\frac{(2.003621)^2-1}{(2.003621)^2+60}\right] - \frac{3}{128}$

$= \frac{3.014497}{128.02899} - \frac{3}{128}$

$= 0.0235454 - 0.0234375$

$= 0.0001079 < +\infty.$

Again for $k = 0$

$= \frac{1}{2}\left[\frac{(\beta^1)^2-1}{(\beta^1)^2+60}\right] - \frac{3}{128}$

Now choose $\boldsymbol{\beta^0 = 4}, \left(\boldsymbol{i.e., \beta^1 = \sqrt{6}}\right)$.

$= \frac{1}{2}\left[\frac{\left(\sqrt{6}\right)^2-1}{\left(\sqrt{6}\right)^2+60}\right] - \frac{3}{128}$

$= \frac{5}{132} - \frac{3}{128}$

$= 0.037879 - 0.0234375$

$= 0.0144415 < +\infty.$

For $k = 1$

$= \frac{1}{2}\left[\frac{(\beta^2)^2-1}{(\beta^2)^2+60}\right] - \frac{3}{128}$

Choose $\beta^1 = \sqrt{6} = 2.449, (i.e., \beta^2 = 2.10927)$.

$= \frac{1}{2}\left[\frac{(2.10927)^2-1}{(2.10927)^2+60}\right] - \frac{3}{128}$

$= \frac{3.44902}{128.89804} - \frac{3}{128}$

$= 0.026758 - 0.0234375$

$= 0.0033205 < +\infty.$

For $k = 2$

$= \frac{1}{2}\left[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}\right] - \frac{3}{128}$

Choose $\beta^2 = 2.10927, (i.e., \beta^3 = 2.027133)$.

$= \frac{1}{2}\left[\frac{(2.027133)^2-1}{(2.027133)^2+60}\right] - \frac{3}{128}$

$= \frac{3.10927}{128.21854} - \frac{3}{128}$

$= 0.0242498 - 0.0234375$

$= 0.0008123 < +\infty.$

For $k = 3$

$= \frac{1}{2}\left[\frac{(\beta^4)^2-1}{(\beta^4)^2+60}\right] - \frac{3}{128}$

Choose $\beta^3 = 2.027133, (i.e., \beta^4 = 2.00677)$.

$= \frac{1}{2}\left[\frac{(2.00677)^2-1}{(2.00677)^2+60}\right] - \frac{3}{128}$

$= \frac{3.027126}{128.05425} - \frac{3}{128}$

$= 0.023639 - 0.0234375$

$= 0.0002015 < +\infty.$

Thus

$$\sum_{k=0}^{+\infty}\left(g(\beta^{k+1}) - \frac{3}{128}\right) = \sum_{k=0}^{+\infty}\left(\frac{1}{2}\left\{\frac{(\beta^{k+1})^2 - 1}{(\beta^{k+1})^2 + 60}\right\} - \frac{3}{128}\right) < +\infty.$$

3.3.4.1 Case 3.1

The sequence $\{\beta^k\}_{k\in N}$ in this case is strictly decreasing.

For $\beta^0 = 3,\ \beta^{k+1} = \sqrt{\beta^k + 2},\ when\ k = 0$, we have

$$\beta^1 = \sqrt{\beta^0 + 2}$$

$$\beta^1 = \sqrt{3 + 2}$$

$$\beta^1 = \sqrt{5}$$

This implies

$$g(\beta^1) - \frac{3}{128} = \frac{1}{2}\left[\frac{(\beta^1)^2 - 1}{(\beta^1)^2 + 60}\right] - \frac{3}{128}$$

$$= \frac{4}{130} - \frac{3}{128}$$

$$= 0.03077 - 0.0234375$$

$$= 0.0073325$$

$when\ k = 1$, we have

$$\beta^2 = \sqrt{\beta^1 + 2}$$

$$\beta^2 = \sqrt{2.23607 + 2}$$

$$\beta^2 = 2.05817152$$

This implies

$$g(\beta^2) - \frac{3}{128} = \frac{1}{2}\left[\frac{(\beta^2)^2 - 1}{(\beta^2)^2 + 60}\right] - \frac{3}{128}$$

$$= \frac{3.23607}{128.47214} - \frac{3}{128}$$

$$= 0.0251889 - 0.0234375$$

$$= 0.00175139$$

Now applying ratio test, we get

$$\frac{g(\beta^2) - \frac{3}{128}}{g(\beta^1) - \frac{3}{128}} = \frac{\frac{1}{2}\left[\frac{(\beta^2)^2 - 1}{(\beta^2)^2 + 60}\right] - \frac{3}{128}}{\frac{1}{2}\left[\frac{(\beta^1)^2 - 1}{(\beta^1)^2 + 60}\right] - \frac{3}{128}} = \frac{0.00175139}{0.0073325} = 0.238853 < 1$$

$when\ k = 2$, we have

$$\beta^3 = \sqrt{\beta^2 + 2}$$

$$\beta^3 = \sqrt{2.0581715 + 2}$$

$$\beta^3 = 2.01449$$

This implies

$$g(\beta^3)-\frac{3}{128}=\frac{1}{2}\left[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}\right]-\frac{3}{128}$$

$$=\frac{3.05817}{128.1163}-\frac{3}{128}$$

$$=0.023870-0.0234375$$

$$=0.0004325$$

Now applying ratio test, we get

$$\frac{g(\beta^3)-\frac{3}{128}}{g(\beta^2)-\frac{3}{128}}=\frac{\frac{1}{2}\left[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}\right]-\frac{3}{128}}{\frac{1}{2}\left[\frac{(\beta^2)^2-1}{(\beta^2)^2+60}\right]-\frac{3}{128}}=\frac{0.0004325}{0.00175139}=0.24695<1$$

$when\ k=3$, we have

$$\beta^4=\sqrt{\beta^3+2}$$

$$\beta^4=\sqrt{2.01449+2}$$

$$\beta^4=2.00362$$

This implies

$$g(\beta^4)-\frac{3}{128}=\frac{1}{2}\left[\frac{(\beta^4)^2-1}{(\beta^4)^2+60}\right]-\frac{3}{128}$$

$$=\frac{3.01449}{128.02899}-\frac{3}{128}$$

$$=0.02355-0.0234375$$

$$=0.0001125$$

Now applying ratio test, we get

$$\frac{g(\beta^4)-\frac{3}{128}}{g(\beta^3)-\frac{3}{128}}=\frac{\frac{1}{2}\left[\frac{(\beta^4)^2-1}{(\beta^4)^2+60}\right]-\frac{3}{128}}{\frac{1}{2}\left[\frac{(\beta^3)^2-1}{(\beta^3)^2+60}\right]-\frac{3}{128}}=\frac{0.0001125}{0.0004325}=0.260116<1$$

Use the ratio test to this end. The sequence $\{\beta^k\}_{k\in N}$ decreases strictly this is how it is

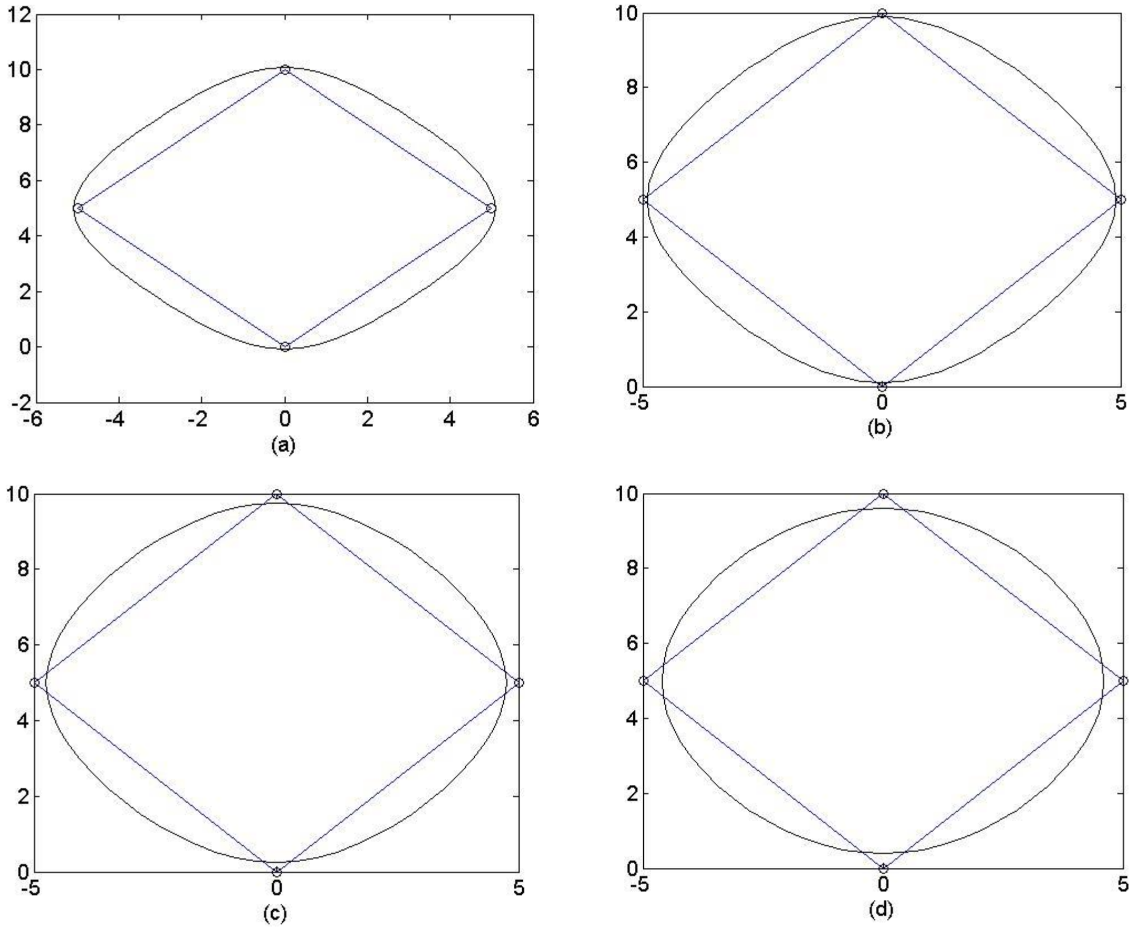
$$\frac{\frac{1}{2}\left[\frac{(\beta^{k+2})^2-1}{(\beta^{k+2})^2+60}\right]-\frac{3}{128}}{\frac{1}{2}\left[\frac{(\beta^{k+1})^2-1}{(\beta^{k+1})^2+60}\right]-\frac{3}{128}}<1$$

The smoothness of (3.6) has therefore been proven.

3.4 Graphical View

We would like to give an example in this section to depict the benefit of the developed technique (3.3). As mentioned in section 3.2, the curves generated tend to approximate the initial discrete polygon of control when $\beta^0 \rightarrow +\infty$.

In Figure 3.1, Generating wide range of C^3-continuous limiting curves for different values of parameters. (a) $\beta^0 = -2$ (b) $\beta^0 = -1$, (c) $\beta^0 = 0$, (d) $\beta^0 = 1$, (e) $\beta^0 = 5$, (f) $\beta^0 = 10$, (g) $\beta^0 = 25$, (h) $\beta^0 = 50$, (i) $\beta^0 = 100$.



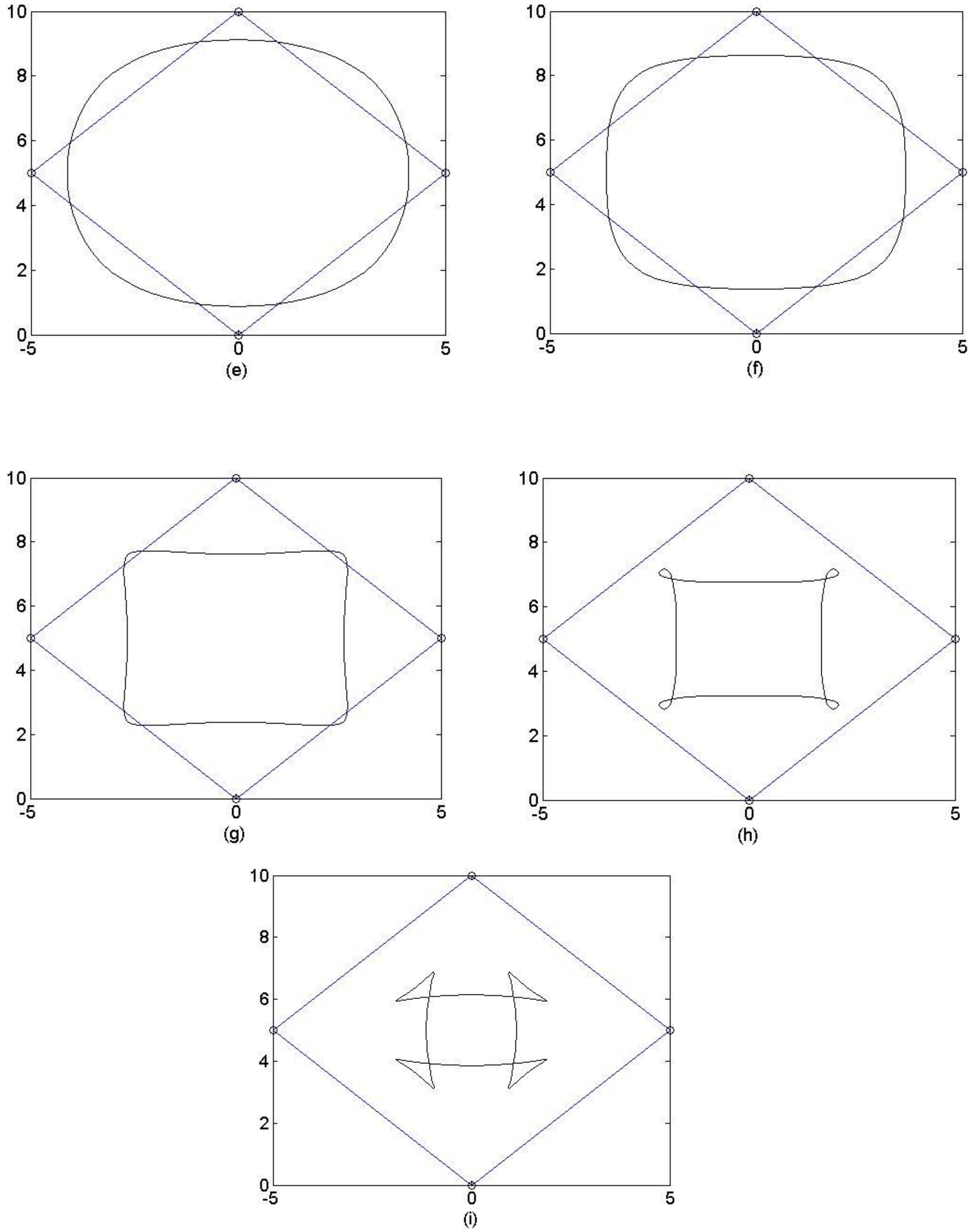


Figure 3.1: Generating wide range of C^3-continuous limiting curves using the proposed scheme (3.3) for different values of parameter. (a) $\beta^0 = -2$, (b) $\beta^0 = -1$, (c) $\beta^0 = 0$, (d) $\beta^0 = 1$, (e) $\beta^0 = 5$, (f) $\beta^0 = 10$, (g) $\beta^0 = 25$, (h) $\beta^0 = 50$, (i) $\beta^0 = 100$.

Conclusion

In this thesis, a new non-stationary binary four-point approximating Subdivision Scheme has been introduced which generates a family of C^3 limiting curves for the wider range of shape control parameters $\beta^0 \in [-2, +\infty)$. The proposed scheme offers considerable flexibility in the construction of C^3 forms in geometric design. The derivative continuity of the technique is investigated by the facts of Dyn and Levin [25].

References

1.1.4 Iterative Method

An iterative method is a mathematical procedure that uses an initial guess to generate a sequence of improving an approximate solution for a class of problem in which the nth approximation is derived from the previous ones.

1.1.5 Refinement

A refinement of a cover is a cover such that every element is a subset of an element.

1.1.6 Sequence

A sequence is an arrangement of numbers written in definite order according to some specific rule.

1.1.7 Increasing sequence

Consider a_n is a sequence of nth term, so if $a_{n+1} > a_n$ for all n, the sequence is said to be increasing. This means that for all n, we have $a_{n+1}/a_n > 1$.

1.1.8 Decreasing sequence

Consider a_n is a sequence of nth term, so if $a_{n+1} < a_n$ for all n, the sequence is said to be decreasing. This means that for all n, we have $a_{n+1}/a_n < 1$.

1.1.9 Control Point

A control point is a member of a set of points used to determine the shape of a spline curve or more generally, a surface or higher dimensional object in computer aided geometric design.

1.1.10 Control Polygon

Control polygon is the sequence of control points in space that is usually used to control an object's shape.

1.1.11 Ratio test

Let $\sum_1^\infty a_n$ be a series of positive terms and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$,

where L is a real number or non-negative numbers

- If $L < 1$, the series $\sum_1^\infty a_n$ converges.
- If $L > 1$, the series $\sum_1^\infty a_n$ diverge.
- If $L = 1$ the test fails to determine convergence or divergence of the series.

1.1.12 Limit curve

In mathematics, a limit is the value approached by a function as the input approaches a specific value.

The two schemes are then equivalent asymptotically. And we can conclude that the e_3^{∞} scheme is C^3.

Since $\quad e_{(3)}^{k} - e_3^{\infty} = 8[g(\beta^{k+1}) - \frac{1}{16}, 2(\frac{1}{16} - g(\beta^{k+1})),\ g(\beta^{k+1}) - \frac{1}{16}]$

$$\left\|e_{(3)}^{k} - e_3^{\infty}\right\|_{\infty} = 8max\left\{2\left|g(\beta^{k+1}) - \frac{1}{16}\right|, 2\left|\frac{1}{16} - g(\beta^{k+1})\right|\right\}$$

$$\left\|e_{(3)}^{k} - e_3^{\infty}\right\|_{\infty} = 16\left|g(\beta^{k+1}) - \frac{1}{16}\right|$$

To prove (2.4), we must prove the smoothness between the series

$$\sum_{k=0}^{+\infty}\left|g(\beta^{k+1}) - \frac{1}{16}\right| \qquad \text{………… (2.5)}$$

which depends on the $g(\beta^{k+1})$ function. Now, since $g(\beta^{k+1})$ is expressed in relation (2.2) in terms of the parameter β^{k+1}, we will study the behaviour of (2.5), since β^{k+1} varies in the interval $[0, +\infty)$. From now on

$\boldsymbol{g(\beta^{k+1}) - \frac{1}{16} = 0 \Leftrightarrow \beta^{k+1} = 3, (i.e., \beta^{0} = 3)}$

$= \frac{1}{2[(\beta^{k+1})^2-1]} - \frac{1}{16}$

$= \frac{1}{2[(3)^2-1]} - \frac{1}{16}$

$= \frac{1}{2(8)} - \frac{1}{16}$

$= \frac{1}{16} - \frac{1}{16}$

$= 0$

$\boldsymbol{g(\beta^{k+1}) - \frac{1}{16} > 0 \Leftrightarrow \beta^{k+1} \in (1, 3)}$ **(i.e.,** $\boldsymbol{\beta^{0} \in [-4, 3)}$**)**

$= \frac{1}{2[(\beta^{k+1})^2-1]} - \frac{1}{16}$

For $k = 0$

$= \frac{1}{2[(\beta^{1})^2-1]} - \frac{1}{16}$

Now choose $\boldsymbol{\beta^{0} = -4, (i.e., \beta^{1} = \sqrt{2})}$

$= \frac{1}{2[(\sqrt{2})^2-1]} - \frac{1}{16}$

$= \frac{1}{2(1)} - \frac{1}{16}$

2.3.3 Case 2:

$\boldsymbol{\beta^0} \in (\mathbf{-5}, \mathbf{3})(\boldsymbol{i.e.}, \boldsymbol{\beta^{k+1}} \in (\mathbf{1}, \mathbf{3}))$.

In this case

$$\left\|e_{(3)}^{k} - e_3^{\infty}\right\|_{\infty} = 16\left[g(\beta^{k+1}) - \frac{1}{16}\right]\backslash$$

$$g(\beta^{k+1}) - \frac{1}{16} = \frac{1}{2}\frac{1}{\left[\left(\beta^{k+1}\right)^2 - 1\right]} - \frac{1}{16}$$

For $k = 0$.

$$g(\beta^1) - \frac{1}{16} = \frac{1}{2}\frac{1}{[(\beta^1)^2 - 1]} - \frac{1}{16}$$

Choose $\boldsymbol{\beta^0} = \mathbf{-4}, (\boldsymbol{i.e.}, \boldsymbol{\beta^1} = \sqrt{\mathbf{2}})$.

$$g(\beta^1) - \frac{1}{16} = \frac{1}{2}\frac{1}{\left[\left(\sqrt{2}\right)^2 - 1\right]} - \frac{1}{16}$$

$$= \frac{1}{2} - \frac{1}{16}$$

$$= 0.5 - 0.0625$$

$$= 0.4375 < +\infty.$$

For $k = 1$

$$g(\beta^2) - \frac{1}{16} = \frac{1}{2[(\beta^2)^2 - 1]} - \frac{1}{16}$$

$\beta^1 = 1.4142, (i.e., \beta^2 = 2.7229)$.

$$= \frac{1}{2[(\ 2.7229\ )^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{12.8284} - \frac{1}{16}$$

$$= 0.07795 - 0.0625$$

$$= 0.01545 < +\infty.$$

For $k = 2$

$$g(\beta^3) - \frac{1}{16} = \frac{1}{2[(\beta^3)^2 - 1]} - \frac{1}{16}$$

$\beta^2 = 2.7229, (i.e., \beta^3 = 2.9535)$.

$$= \frac{1}{2[(2.9535)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{15.4463} - \frac{1}{16}$$

$$= 0.0647 - 0.0625$$

$$= 0.0022 < +\infty.$$

For $k = 3$

$$g(\beta^3) - \frac{1}{16} = \frac{1}{2[(\beta^4)^2 - 1]} - \frac{1}{16}$$

$\beta^3 = 2.9535, (i.e., \beta^4 = 2.9922).$

$$= \frac{1}{2[(2.9922)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{15.9065} - \frac{1}{16}$$

$$= 0.0629 - 0.0625$$

$$= 0.0004 < +\infty.$$

Again for $k = 0$

$$g(\beta^1) - \frac{1}{16} = \frac{1}{2}\frac{1}{[(\beta^1)^2 - 1]} - \frac{1}{16}$$

Choose $\boldsymbol{\beta^0 = -2}, (\boldsymbol{i.e.}, \boldsymbol{\beta^1 = 2})$.

$$= \frac{1}{2}\frac{1}{[(2)^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{6} - \frac{1}{16}$$

$$= 0.16666 - 0.0625$$

$$= 0.1041660 < +\infty.$$

For $k = 1$

$$g(\beta^2) - \frac{1}{16} = \frac{1}{2[(\beta^2)^2 - 1]} - \frac{1}{16}$$

$\beta^1 = 2, (i.e., \beta^2 = 2.82843).$

$$= \frac{1}{2[(\ 2.82843\ )^2 - 1]} - \frac{1}{16}$$

$$= \frac{1}{14.000} - \frac{1}{16}$$

$$= 0.07143 - 0.0625$$

$$= 0.008929 < +\infty.$$

Chapter # 3

A non-stationary four-point subdivision technique

3.1 Stationary four-point subdivision technique

Kim *et al.* [13] proposed a binary subdivision scheme with four points that generates a smooth C^3-continuous limiting curve. Given the set of control points $q^0 = \{q_i{}^0\}_{i\in Z}$ at level 0, the binary four point SS for the design of curves generates a new set of control points $\{q_i{}^{k+1}\}_{i\in Z}$ at level k+1 using the following subdivision rules

$$q_{2i}^{k+1} = -\beta_0^k q_{i-2}^k + \beta_1^k q_{i-1}^k + \beta_2^k q_i^k + \beta_3^k q_{i+1}^k - \beta_4^k q_{i+2}^k$$

$$q_{2i+1}^{k+1} = -\frac{1}{16} q_{i-1}^k + \frac{9}{16} q_i^k + \frac{9}{16} q_{i+1}^k - \frac{1}{16} q_{i+2}^k. \qquad \ldots\ldots\ldots\ldots(3.1)$$

Where $q^0 = \{q_i^0\}_{i\in Z}$ is the set of initial control point at level 0 and the mask of the scheme, the relationship $\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$ shall be met. The scheme coefficients (3.1) are

$$\beta_0 = \frac{3}{128}, \beta_1 = \frac{12}{128}, \beta_2 = \frac{110}{128}, \beta_3 = \frac{12}{128}, \beta_4 = \frac{3}{128}.$$

They found that the scheme is C^1-continuous when and the scheme is C^2-continuous when $-0.12 < \beta < 0.21$ and the scheme is C^3-continuous when $-0.88 < \beta < 0.13$. For the range of $-0.88 < \beta < 0.13$, the proposed scheme is non-stationary scheme.

3.2 Non-stationary four-point subdivision technique

The refining rules of the binary non-stationary SS of four points are defined as

$$q_{2i}^{k+1} = -\beta_0^k q_{i-2}^k + \beta_1^k q_{i-1}^k + \beta_2^k q_i^k + \beta_3^k q_{i+1}^k - \beta_4^k q_{i+2}^k$$

$$q_{2i+1}^{k+1} = -\frac{1}{16} q_{i-1}^k + \frac{9}{16} q_i^k + \frac{9}{16} q_{i+1}^k - \frac{1}{16} q_{i+2}^k. \qquad \ldots\ldots\ldots\ldots(3.2)$$

Where $q^0 = \{q_i^0\}_{i\in Z}$ is the set of initial control point at level 0 and the mask of the scheme, the relationship $\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$ shall be met.

The binary non-stationary four point subdivision scheme (3.2) is counter part of stationary scheme [13]. The mask of the scheme are given by

$$\beta_0^k = g(\beta^{k+1}) = \beta_4^k, \quad \beta_1^k = 4g(\beta^{k+1}) = \beta_3^k, \quad \beta_2 = 1 - 6g(\beta^{k+1});$$

$$g(\beta^{k+1}) = \frac{1}{2}\left[\frac{(\beta^{k+1})^2 - 1}{(\beta^{k+1})^2 + 60}\right] \qquad \ldots\ldots\ldots\ldots(3.3)$$

With

$$\beta^{k+1} = \sqrt{\beta^k + 2}, \qquad and \qquad \beta^0 \in [-2, +\infty). \qquad \ldots\ldots\ldots\ldots(3.4)$$

$$m^k = \left[-g(\beta^{k+1}), -\frac{1}{16}, 4g(\beta^{k+1}), \frac{9}{16}, 1-6g(\beta^{k+1}), \frac{9}{16}, 4g(\beta^{k+1}), -\frac{1}{16}, -g(\beta^{k+1})\right]$$

Then it turns out its first divided difference masks are

$$e^k_{(1)} = 2\begin{bmatrix} -g(\beta^{k+1}), \left(g(\beta^{k+1}) - \frac{1}{16}\right), \left(3g(\beta^{k+1}) + \frac{1}{16}\right), \left(\frac{1}{2} - 3g(\beta^{k+1})\right), \\ \left(\frac{1}{2} - 3g(\beta^{k+1})\right), \left(3g(\beta^{k+1}) + \frac{1}{16}\right), \left(g(\beta^{k+1}) - \frac{1}{16}\right), -g(\beta^{k+1}) \end{bmatrix}$$

Then it turns out its 2nd divided difference masks are

$$e^k_{(2)} = 4\begin{bmatrix} -g(\beta^{k+1}), \left(2g(\beta^{k+1}) - \frac{1}{16}\right), \left(g(\beta^{k+1}) + \frac{1}{8}\right), \left(\frac{3}{8} - 4g(\beta^{k+1})\right), \\ \left(g(\beta^{k+1}) + \frac{1}{8}\right), \left(2g(\beta^{k+1}) - \frac{1}{16}\right), -g(\beta^{k+1}) \end{bmatrix}$$

Then it turns out its 3rd divided difference masks are

$$e^k_{(3)} = 8\begin{bmatrix} -g(\beta^{k+1}), \left(3g(\beta^{k+1}) - \frac{1}{16}\right), \left(-2g(\beta^{k+1}) + \frac{3}{16}\right), \\ \left(-2g(\beta^{k+1}) + \frac{3}{16}\right), \left(3g(\beta^{k+1}) - \frac{1}{16}\right), -g(\beta^{k+1}) \end{bmatrix}$$

The application of Remark 2 now provides

$$e^\infty_{(3)} = \lim_{k\to+\infty} e^k_3 = 8\left[-\frac{3}{128}, \frac{1}{128}, \frac{18}{128}, \frac{18}{128}, \frac{1}{128}, -\frac{3}{128}\right]$$

This is just the coefficients of the third divided differences of the stationary technique with coefficients in equation (3.1). In this case, the stationary technique is C^3-continuous, the technique associated with e^∞_3 will be C^3 smooth. If it is as

$$\sum_{k=0}^{+\infty} \left\| e^k_{(3)} - e^\infty_3 \right\|_\infty < +\infty. \qquad \text{………(3.5)}$$

The two techniques are then equivalent asymptotically. And one can conclude this that the e^∞_3 of the technique is C^3, since then

$$e^k_{(3)} - e^\infty_3 = 8[(-2g(\beta^{k+1}) + \tfrac{6}{128}), -3\,(-2g(\beta^{k+1}) + \tfrac{6}{128}), 2(-2g(\beta^{k+1}) + \tfrac{6}{128})]$$

$$\left\| e^k_{(3)} - e^\infty_3 \right\|_\infty = 8max\left\{\left|-2g\left(\beta^{k+1}\right) + \tfrac{6}{128}\right|, |-3| \left|-2g\left(\beta^{k+1}\right) + \tfrac{6}{128}\right|, |2|\left|-2g\left(\beta^{k+1}\right) + \tfrac{6}{128}\right|\right\}$$

$$= 24\left|-2g(\beta^{k+1}) + \tfrac{6}{128}\right|$$

$$= 48\left|\tfrac{3}{128} - g(\beta^{k+1})\right|$$

Now we are proving the series smoothness

$$\sum_{k=0}^{+\infty} \left|\frac{3}{128} - g(\beta^{k+1})\right|. \qquad \text{………..(3.6)}$$

Conclusion

In this thesis, a new non-stationary binary four-point approximating subdivision scheme has been introduced which generates a family of C^3 limiting curves for the wider range of shape control parameters $\beta^0 \in [-2, +\infty)$. The proposed scheme offers considerable flexibility in the construction of C^3 forms in geometric design. The derivative continuity of the technique is investigated by the facts of Dyn and Levin [25].