HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR HARMONICALLY RELATIVE PREINVEX FUNCTIONS

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2016-M.PHIL-APP-MATH-25

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LAHORE – PAKISTAN

2018

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This Thesis is submitted to the Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan, for the partial fulfillment of the requirements for the award of the degree of

Master of Philosophy (M. Phil)

In

APPLIED MATHEMATICS

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بسم الله الرحمن الرحيم

Read! In the Name of your Lord, Who has created (all that exists),

Has created man from a clot (a piece of thick coagulated blood).

Read! And your Lord is the Most Generous,

Who has taught (the writing) by the pen [the first person to write was Prophet Idrees (Enoch)],

Has taught man that which he knew not.

Al-Quran

DEDICATION

This Work is Dedicated to the Most
Precious Asset of Our Life,

"Our Parents,"

Whose Prayers and Love Have Always
Flattened the Thorny Road of Our Life.

Acknowledgement

Up and above anything else, all praises are due to **Almighty Allah** Alone, the Omnipotent, and the Omnipresent. The most Merciful and the most Beneficent and after Almighty Allah to Holy Prophet Hazrat Muhammad (SAW) the Most perfect and Exalted, Who is forever a source of guidance and knowledge for humanity as a whole.

Profound testimonies and acknowledgement are due to my supervisor, **Dr. Sabir Hussain** , who took tremendous pains in the completion of this Thesis. It was impossible for me to complete this work without his constant advice encouragement, generous help, guidance and valuable comments. I have gained tremendously from the vast reservoir of knowledge which he possess and I feel that these few lines of acknowledgement reflect only a fraction of gratitude which I owe to him. As someone said "Accept my thoughts for thanks: I have no words".

I would also render my turgidities to my respected teachers for their continuous help and encouragement especially for **Prof. Dr. Muhammad Mushtaq** for his valuable suggestions.

Of course, I am grateful to parents my, brothers, sisters for their patience and love. Without them this work would never have come into existence (literally). Their love, moral support and prayers are great source of strength for me in every field of life. Without their sacrifices and encouragement, the present work would have been a merry dream.

The Author

ABSTRACT

Inequalities plays a significant and wide spread role in the evolution of many fields of mathematics. In the growth of finite difference, integral and differential equations, there is far-reaching part of inequalities and their explicit estimates. The field of finite difference and integral inequalities along with explicit estimates have great efficacy in the study of qualitative properties of solutions of numerous types of finite difference equations.

Hermite-Hadamard integral inequality is one of the famous inequality used for harmonically convex functions. By using the concept of harmonically relative preinvex functions we introduce several new upper bounds of Hermite-Hadamard type integral inequalities for harmonically relative preinvex functions and their different types such as s-harmonic preinvex functions, s-harmonic Godunova Levin functions and harmonic P-preinvex functions.

Chapter 1:

This chapter deals with the introduction, history and background of inequalities.

Chapter 2

In the second chapter, we studied some new upper bounds of Hermite-Hadamard type inequalities for harmonically convex functions.

Chapter 3:

This chapter deals with several new upper bounds of Hermite-Hadamard type integral inequalities for harmonically relative preinvex functions and their different types.

CONTENTS

CHAPTER 1

LITERATURE REVIEW AND BACKGROUND

1.1 Introduction

The beginning of the theory of convexity can be traced back to the end of 19^{th} century. In this time and with emergence of calculus the touch of inequalities seemingly became essential. Convexity and generalized convexity play fundamental role in mathematical economics, engineering, management sciences and optimization theory. Consequently, the research on convexity and generalized convexity is one of the most significant aspects in mathematical programming. The theory of convexity has been extended in numerous directions using advanced ideas and techniques [31]. It plays an important role in other fields of mathematics: complex analysis, functional analysis, discrete mathematics, calculus of variations, partial differential equations, graph theory, algebraic geometry, coding theory and many other areas. Several inequalities have been obtained for convex function but a very well-known is the Hermite-Hadamard inequality.

Hermite-Hadamard inequality was discovered by Ch. Hermite [10] in 1883 and rediscovered by J. Hadamard [8] in 1893. Hermite-Hadamard inequalities for convex functions and their several forms exist in literature [1, 6, 15, 18, 21, 29].

The generalization of convexity is the invexity, many researchers have done work on it. Hanson [9] investigate and introduced the invex functions. Ben-Israel and Mond [5] worked on invex set and preinvex functions. Pini [32] investigated another class of generalized invex functions, named as preinvex functions. Mohan and Neogy [14] established some properties of generalized preinvex functions. Noor [26] introduced some Hermite-Hadamard type inequalities for preinvex functions. Various integral inequalities for preinvex functions have been established recently, see [26]. Iscan [7] introduced the concept of harmonically convex functions.

Noor et. al. [22] investigate a class of preinvex functions with respect to an arbitrary function h, which is said to be relative preinvex functions. He also introduced the class of relative harmonic functions with respect to an arbitrary nonnegative function h and established a innovative class of convex function with respective to an arbitrary nonnegative function h, which is known as relative harmonic preinvex functions [27]. We also obtain divers classes of harmonic convex and harmonic preinvex functions such as Breckner type of s-harmonic preinvex functions, Godunova levin type of s-harmonic preinvex functions and harmonic P-preinvex functions.
Now we recall some basic results and concepts [26, 30].

1.1.1 Convex Set

A set $J \in \mathbb{R}^n$ is known as convex set, if

$$(1-t)m + tn \in J, \quad t \in [0,1], \quad \forall\ m, n \in J.$$

1.1.2 Convex Function

Let $J \subseteq \mathbb{R}^n$ be convex set, A function $f : J \longrightarrow \mathbb{R}$ is known as convex, if

$$f((1-t)m + tn) \leq (1-t)f(m) + tf(n), \quad t \in [0,1], \quad \forall\ m, n \in J. \tag{1.1}$$

By changing the sign of inequality it becomes a concave function.

Remark 1 *If $t = \frac{1}{2}$ in (1.1), then we get*

$$f\left(\frac{m+n}{2}\right) \leq \frac{f(m)+f(n)}{2} \quad \forall\ m, n \in J.$$

which is known as Jensen convex function.

1.1.3 Harmonic Convex Set

A set $J \subseteq \mathbb{R} \setminus \{0\}$ is known as harmonic convex set, if

$$\frac{mn}{(1-t)m + tn} \in J, \quad t \in [0,1], \quad \forall\ m, n \in J.$$

1.1.4 Harmonic Convex Function

A function $f : J \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is known as harmonic convex function, if

$$f\left(\frac{mn}{(1-t)m + tn}\right) \leq tf(m) + (1-t)f(n), \quad t \in [0,1], \quad \forall\ m, n \in J.$$

1.1.5 Relative Convex Function

A function $f : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is known as relative convex function with respect to arbitrary function h, where $h : [0,1] \subseteq I \longrightarrow \mathbb{R}$ is a non-negative function, if

$$f((1-t)m + tn) \leq h(1-t)f(m) + h(t)f(n), \quad t \in [0,1], \ \forall\ m, n \in J.$$

Remark 2 *If $h(t) = t$, then the relative convex function becomes convex function.*

1.1.6 Relative Harmonic Convex Function

A function $f : J \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ on harmonic convex set J is known as relative harmonic convex function with respect to arbitrary function h, where $h : [0,1] \subseteq I \longrightarrow \mathbb{R}$ is a non-negative function, if

$$f\left(\frac{mn}{(1-t)m+tn}\right) \leq h(t)f(m) + h(1-t)f(n), \quad t \in [0,1], \ \forall \ m,n \in J.$$

1.1.7 Invex Set

Assuming that J be a non-empty closed set in $\mathbb{R}^n$, let $\eta(.,.) : J \times J \longrightarrow \mathbb{R}^n$ be a continuous bifunction. Then J is known as invex with respect to $\eta(.,.)$, if

$$m + t\eta(n,m) \in J, \quad t \in [0,1], \quad \forall \ m,n \in J.$$

Remark 3 *If $\eta(n,m) = n - m$, then invex set J becomes convex set. Clearly, every convex set is an invex set but the converse is not true.*

1.1.8 Preinvex Function

Let $J \subseteq \mathbb{R}^n$ be invex set, A function $f : J \longrightarrow \mathbb{R}$ is known as preinvex with respect to bifunction $\eta(.,.)$, if

$$f(m + t\eta(n,m)) \leq (1-t)f(m) + tf(n), \quad t \in [0,1], \quad \forall \ m,n \in J.$$

1.1.9 Harmonic Invex Set

A set $K = [m, m+\eta(n,m)] \subseteq \mathbb{R} \setminus \{0\}$ is known as harmonic invex set with respect to the bifunction $\eta(.,.)$, if

$$\frac{m(m+\eta(n,m))}{m+(1-t)\eta(n,m)} \in K, \quad t \in [0,1], \quad \forall \ m,n \in K.$$

Remark 4 *If $\eta(n,m) = n - m$, then harmonic invex set K becomes harmonic convex set. So, every harmonic set is an invex set but the converse is not true.*

1.1.10 Harmonic Preinvex Function

A function $f : K = [m, m+\eta(n,m)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is known as harmonic preinvex function with respect to bifunction $\eta(.,.)$, if

$$f\left(\frac{m(m+\eta(n,m))}{m+(1-t)\eta(n,m)}\right) \leq (1-t)f(m) + tf(n), \quad t \in [0,1], \quad \forall \ m,n \in K.$$

1.1.11 Hermite-Hadamard Inequality for Convex Function

A function $f : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is convex function on $J = [m,n]$ if and only if , f satisfies the inequality

$$f\left(\frac{m+n}{2}\right) \leq \frac{1}{n-m}\int_m^n f(x)dx \leq \frac{f(m)+f(n)}{2}.$$

which is known as Hermite-Hadamard inequality for convex function.

1.1.12 Hermite-Hadamard Inequality for Harmonically Convex Function

A function $f : J \subseteq (0,\infty) \longrightarrow \mathbb{R}$ is harmonically convex function on the interval $J = [m,n]$ and $f \in L[m,n]$, where $m,n \in J$ with $m < n$, if and only if , f satisfies the inequality

$$f\left(\frac{2mn}{m+n}\right) \leq \frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx \leq \frac{f(m)+f(n)}{2}.$$

which is known as Hermite-Hadamard inequality for harmonically convex function.

1.1.13 Hermite-Hadamard-Noor type Inequality

A function f is prinvex function if and only if, f satisfies the inequality of the type ,$\forall\ m,n \in [m, m+\eta(n,m)]$

$$f\left(\frac{2m+\eta(n,m)}{2}\right) \leq \frac{1}{\eta(n,m)}\int_m^{n+\eta(n,m)} f(x)dx \leq \frac{f(m)+f(n)}{2}.$$

which is known as Hermite-Hadamard-Noor inequality .

Remark 5 *If $\eta(n,m) = n-m$, then the Hermite-Hadamard-Noor inequality becomes the Hermite-Hadamard inequality for convex function.*

1.1.14 Hypergeometric Function

The ${}_2F_1[r,s,t,x]$ is hypergeometric function which is shown as follows

$${}_2F_1[r,s,t,x] = \sum_{p=0}^{\infty} \frac{(r)_p(s)_p}{(t)_p}\frac{x^p}{p!}; \qquad |x| < 1$$

It is not defined if t equals a non-positive integer. Here $(v)_p$ be Pochhammer symbol, which is given by

$$(v)_p = \begin{cases} 1, & p = 0 \\ v(v+1)...(v+p-1), & p > 0 \end{cases}$$

1.1.15 Beta Function

The beta function is special function, also known as the Euler integral of the first kind is denoted as

$$B(m,n) = \int_0^1 z^{m-1}(1-z)^{n-1}dz = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}; \quad where\ \ m,n\ \ are\ real\ numbers.$$

CHAPTER 2

HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS

2.1 Introduction

The Hermite-Hadamard is most well-known inequalties related to integral mean of convex functions. The concept of convexity for functions have been generalized and extended in many directions and in multiple forms, a wide range of generalizations have been established within the literature using concept of convexity.

Imdat Işcan [7] worked on defination of harmonically convex function and also gave the Hermite-Hadamard inequality for harmonically convex function. Further, Aslam Noor [18, 21] and Amer Latif [11, 12] established some new results in this direction to our best knowledge. Now, we derive some Hermite-Hadamard type inequalities for differentiable harmonically convex functions.

2.2 Main Results

Lemma 1 *.[30] Assuming that* $f : M = [u, v] \subseteq (0, \infty) \longrightarrow \mathbb{R}$ *is differentiable on* $M^o = (u, v)$ *of* M*. If* $f \in L_1[u, v]$*,* u, v *in* M *with* $u < v$*, and* $\lambda, \alpha \in [0, 1]$*, then the following equality holds for*

$t \in [0,1]$ *and* $A_t = (1-t)u + tv$, *we have*

$$\begin{aligned}\Psi_f(\lambda,\alpha,u,v) &:= \{(1-\alpha)\lambda f(u) + \alpha\lambda f(v)\} - (\lambda-1) f\left(\frac{uv}{A_{1-\alpha}}\right) - \frac{uv}{v-u}\int_u^v \frac{f(x)}{x^2}dx \\ &= uv(u-v)\left[\int_0^{1-\alpha} \frac{t-\alpha\lambda}{{A_t}^2} f'\left(\frac{uv}{A_t}\right)dt + \int_{1-\alpha}^1 \frac{t-(1+\lambda(\alpha-1))}{{A_t}^2} f'\left(\frac{uv}{A_t}\right)dt\right].\end{aligned}$$

Proof. Let

$$\begin{aligned}I_1 &= \int_0^{1-\alpha} \frac{t-\alpha\lambda}{{A_t}^2} f'\left(\frac{uv}{A_t}\right)dt \\ &= -\int_0^{1-\alpha} \frac{t-\alpha\lambda}{((1-t)u+tv)^2}\frac{-uv(v-u)}{((1-t)u+tv)^2}\frac{((1-t)u+tv)^2}{uv(v-u)} f'\left(\frac{uv}{(1-t)u+tv}\right)dt \\ &= -\int_0^{1-\alpha} \frac{t-\alpha\lambda}{uv(v-u)} df\left(\frac{uv}{(1-t)u+tv}\right) \\ &= -\frac{1}{uv(v-u)}\int_0^{1-\alpha} (t-\alpha\lambda) df\left(\frac{uv}{(1-t)u+tv}\right) \\ &= -\frac{1}{uv(v-u)}\left[(t-\alpha\lambda) f\left(\frac{uv}{(1-t)u+tv}\right)\Big|_0^{1-\alpha} - \int_0^{1-\alpha} f\left(\frac{uv}{(1-t)u+tv}\right)dt\right] \\ &= -\frac{1}{uv(v-u)}\left[\{(1-\alpha)-\alpha\lambda\} f\left(\frac{uv}{A_{1-\alpha}}\right) + \alpha\lambda f(v) - \int_0^{1-\alpha} f\left(\frac{uv}{(1-t)u+tv}\right)dt\right] \quad (2.1)\end{aligned}$$

Now, let

$$\begin{aligned}I_2 &= \int_{1-\alpha}^1 \frac{t-(1+\lambda(\alpha-1))}{(A_t)^2} f'\left(\frac{uv}{A_t}\right)dt \\ &= -\int_{1-\alpha}^1 \frac{t-(1+\lambda(\alpha-1))}{((1-t)u+tv)^2}\frac{-uv(v-u)}{((1-t)u+tv)^2}\frac{((1-t)u+tv)^2}{uv(v-u)} f'\left(\frac{uv}{(1-t)u+tv}\right)dt \\ &= -\int_{1-\alpha}^1 \frac{t-(1+\lambda(\alpha-1))}{uv(v-u)} df\left(\frac{uv}{(1-t)u+tv}\right) \\ &= -\frac{1}{uv(v-u)}\int_{1-\alpha}^1 t-(1+\lambda(\alpha-1)) df\left(\frac{uv}{(1-t)u+tv}\right) \\ &= -\frac{1}{uv(v-u)}\left[t-(1+\lambda(\alpha-1)) f\left(\frac{uv}{(1-t)u+tv}\right)\Big|_{1-\alpha}^1 - \int_{1-\alpha}^1 f\left(\frac{uv}{(1-t)u+tv}\right)dt\right] \\ &= -\frac{1}{uv(v-u)}\left[-\{(-\alpha)+\lambda(1-\alpha)\} f\left(\frac{uv}{A_{1-\alpha}}\right)\right. \\ &\quad \left. +\lambda(1-\alpha)f(u) - \int_{1-\alpha}^1 f\left(\frac{uv}{(1-t)u+tv}\right)dt\right]. \quad (2.2)\end{aligned}$$

Adding Equations (2.1) and (2.2), we have

$$\begin{aligned}\Psi_f(\lambda,\alpha,u,v) &= -\frac{1}{uv(v-u)}\left[\{(1-\alpha)-\alpha\lambda-(-\alpha)-\lambda(1-\alpha)\} f\left(\frac{uv}{A_{1-\alpha}}\right) + \lambda(1-\alpha)f(u)\right. \\ &\quad \left. +\alpha\lambda f(v) - \int_0^{1-\alpha} f\left(\frac{uv}{(1-t)u+tv}\right)dt - \int_{1-\alpha}^1 f\left(\frac{uv}{(1-t)u+tv}\right)dt\right] \\ &= -\frac{1}{uv(v-u)}\left[\lambda\{(1-\alpha)f(u)+\alpha f(v)\} + (1-\lambda) f\left(\frac{uv}{A_{1-\alpha}}\right)\right. \\ &\quad \left. -\int_0^1 f\left(\frac{uv}{(1-t)u+tv}\right)dt\right]. \quad (2.3)\end{aligned}$$

Setting $x = \frac{uv}{(1-t)u+tv}$, so that $dx = \frac{-uv(v-u)}{((1-t)u+tv)^2}dt$

For $0 \le t \le 1$, we have $v \le t \le u$ and hence (2.3) becomes

$$\begin{aligned}\Psi_f(\lambda,\alpha,u,v) &=: \lambda\{(1-\alpha)f(u)+\alpha f(v)\}+(1-\lambda)f\left(\frac{uv}{A_{1-\alpha}}\right)-\frac{uv}{v-u}\int_u^v \frac{f(x)}{x^2}dx \\ &= uv(u-v)\left[\int_0^{1-\alpha}\frac{t-\alpha\lambda}{{A_t}^2}f'\left(\frac{uv}{A_t}\right)dt+\int_{1-\alpha}^1 \frac{t-(1+\lambda(\alpha-1))}{{A_t}^2}f'\left(\frac{uv}{A_t}\right)dt\right].\end{aligned}$$

□

Remark 6 **(a)** *If $\lambda=0$, $\alpha=\frac{1}{2}$, then Lemma 1 reduces to the following result*

$$f\left(\frac{uv}{A_{\frac{1}{2}}}\right)-\frac{uv}{(v-u)}\int_u^v\frac{f(x)}{x^2}dx = uv(u-v)\left[\int_0^{\frac{1}{2}}\frac{t}{(A_t)^2}f'\left(\frac{uv}{A_t}\right)dt+\int_{\frac{1}{2}}^1\frac{(t-1)}{(A_t)^2}f'\left(\frac{uv}{A_t}\right)dt\right]$$

$$f\left(\frac{2vu}{u+v}\right)-\frac{uv}{(v-u)}\int_u^v\frac{f(x)}{x^2}dx = uv(u-v)\left[\int_0^{\frac{1}{2}}\frac{t}{(A_t)^2}f'\left(\frac{uv}{A_t}\right)dt+\int_{\frac{1}{2}}^1\frac{(t-1)}{(A_t)^2}f'\left(\frac{uv}{A_t}\right)dt\right]$$

$$\frac{uv}{(v-u)}\int_u^v\frac{f(x)}{x^2}dx-f\left(\frac{2uv}{u+v}\right)= uv(v-u)\left[\int_0^{\frac{1}{2}}\frac{t}{(A_t)^2}f'\left(\frac{uv}{A_t}\right)dt+\int_{\frac{1}{2}}^1\frac{(t-1)}{(A_t)^2}f'\left(\frac{uv}{A_t}\right)dt\right] \quad (2.4)$$

(b) *If $\lambda=1$, $\alpha=\frac{1}{2}$, then Lemma 1 reduces to the following result*

$$\begin{aligned}\frac{f(u)+f(v)}{2}-\frac{uv}{v-u}\int_u^v\frac{f(x)}{x^2}dx &= uv(u-v)\left[\int_0^{\frac{1}{2}}\frac{t-\frac{1}{2}}{{A_t}^2}f'\left(\frac{uv}{A_t}\right)dt+\int_{\frac{1}{2}}^1\frac{(t-1)+\frac{1}{2}}{{A_t}^2}f'\left(\frac{uv}{A_t}\right)dt\right] \\ &= uv(u-v)\left[\int_0^{\frac{1}{2}}\frac{t-\frac{1}{2}}{{A_t}^2}f'\left(\frac{uv}{A_t}\right)dt+\int_{\frac{1}{2}}^1\frac{t-\frac{1}{2}}{{A_t}^2}f'\left(\frac{uv}{A_t}\right)dt\right] \\ &= uv(v-u)\left[\int_0^{\frac{1}{2}}\frac{(\frac{1}{2}-t)}{{A_t}^2}f'\left(\frac{uv}{A_t}\right)dt+\int_{\frac{1}{2}}^1\frac{(\frac{1}{2}-t)}{{A_t}^2}f'\left(\frac{uv}{A_t}\right)dt\right] \\ &= uv(v-u)\int_0^1\frac{(\frac{1}{2}-t)}{{A_t}^2}f'\left(\frac{uv}{A_t}\right)dt\end{aligned}$$

Now, we establish some new integral inequalities of Hermite-Hadamard type for harmonically convex functions.

Theorem 1 *[30] Assuming that $f : M=[u,v]\subseteq\mathbb{R}_+=(0,\infty)\longrightarrow\mathbb{R}$ is differentiable on the interior, $M^o=(u,v)$ of M where $f'\in L[u,v]$ for u,v in M with $u<v$ and $0\le\alpha,\lambda\le 1$. If $|f'|^\mu$ is harmonically convex on M for $\mu\in(1,\infty)$ with $\frac{1}{\gamma}+\frac{1}{\mu}=1$, we have*

(a) *If $\alpha\lambda\le 1-\alpha\le 1+\lambda(\alpha-1)$, then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,u,v)| &\le uv(v-u)[(m_1(\lambda,\alpha,u,v)+m_2(\lambda,\alpha,u,v))^{\frac{1}{\gamma}}\ \{(m_7(\lambda,\alpha,u,v) \\ &\quad +m_8(\lambda,\alpha,u,v))|f'(u)|^\mu+(m_9(\lambda,\alpha,u,v)+m_{10}(\lambda,\alpha,u,v))|f'(v)|^\mu\}^{\frac{1}{\mu}} \\ &\quad +(m_5(\lambda,\alpha,u,v)+m_6(\lambda,\alpha,u,v))^{\frac{1}{\gamma}}\{(m_{15}(\lambda,\alpha,u,v)+m_{16}(\lambda,\alpha,u,v)) \\ &\quad \times|f'(u)|^\mu+(m_{17}(\lambda,\alpha,u,v)+m_{18}(\lambda,\alpha,u,v))|f'(v)|^\mu\}^{\frac{1}{\mu}}].\end{aligned}$$

(b) *If $\alpha\lambda\le 1+\lambda(\alpha-1)\le 1-\alpha$, then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,u,v)| &\le uv(v-u)[(m_1(\lambda,\alpha,u,v)+m_2(\lambda,\alpha,u,v))^{\frac{1}{\gamma}}\ \{(m_7(\lambda,\alpha,u,v) \\ &\quad +m_8(\lambda,\alpha,u,v))|f'(u)|^\mu+(m_9(\lambda,\alpha,u,v)+m_{10}(\lambda,\alpha,u,v))|f'(v)|^\mu\}^{\frac{1}{\mu}} \\ &\quad +(m_4(\lambda,\alpha,u,v))^{\frac{1}{\gamma}}\{(m_{13}(\lambda,\alpha,u,v)|f'(u)|^\mu+m_{14}(\lambda,\alpha,u,v)|f'(v)|^\mu)\}^{\frac{1}{\mu}}].\end{aligned}$$

(c) *If* $1-\alpha \leq \alpha\lambda \leq 1+\lambda(\alpha-1)$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,u,v)| \leq\ & uv(v-u)[(m_3(\lambda,\alpha,u,v))^{\frac{1}{\gamma}}\{(m_{11}(\lambda,\alpha,u,v)|f'(u)|^{\mu}+m_{12}(\lambda,\alpha,u,v)\\ &\times|f'(v)|^{\mu})\}^{\frac{1}{\mu}}+(m_5(\lambda,\alpha,u,v)\ +\ m_6(\lambda,\alpha,u,v))^{\frac{1}{\gamma}}\{(m_{15}(\lambda,\alpha,u,v)\\ &+m_{16}(\lambda,\alpha,u,v))|f'(u)|^{\mu}+(m_{17}(\lambda,\alpha,u,v)+m_{18}(\lambda,\alpha,u,v))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].\end{aligned}$$

Proof. By using Lemma 1 and power mean integral inequality, we have

$$\begin{aligned}|\Psi_f(\lambda,\alpha,u,v)| \leq\ & uv(v-u)\left[\int_0^{1-\alpha}\frac{|t-\alpha\lambda|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|dt+\int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|dt\right]\\ \leq\ & uv(v-u)\left(\int_0^{1-\alpha}\frac{|t-\alpha\lambda|}{(A_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{1-\alpha}\frac{|t-\alpha\lambda|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\\ &+\left(\int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}dt\right)^{1-\frac{1}{\mu}}\\ &\left(\int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}} \qquad (2.5)\end{aligned}$$

(a) (i) If $\alpha\lambda \leq 1-\alpha$, then

$$\begin{aligned}\int_0^{1-\alpha}\frac{|t-\alpha\lambda|}{(A_t)^2}dt &= \int_0^{\alpha\lambda}\frac{-(t-\alpha\lambda)}{(A_t)^2}dt+\int_{\alpha\lambda}^{1-\alpha}\frac{t-\alpha\lambda}{(A_t)^2}dt\\ &= m_1(\lambda,\alpha,u,v)+m_2(\lambda,\alpha,u,v). \qquad (2.6)\end{aligned}$$

where

$$\begin{aligned}m_1(\lambda,\alpha,u,v) &:= \frac{-u\alpha\lambda+v\alpha\lambda+u\log(\frac{-u}{u(-1+\alpha\lambda)-v\alpha\lambda})}{u(u-v)^2}\\ m_2(\lambda,\alpha,u,v) &:= \frac{u-u\alpha\lambda+v\alpha\lambda+(v+u\alpha-v\alpha)\log(v(-1+\alpha)-u\alpha)}{(u-v)^2(v+u\alpha-v\alpha)}\\ &-\frac{1+\log(u(-1+\alpha\lambda)-v\alpha\lambda)}{(u-v)^2}\end{aligned}$$

(ii) If $\alpha\lambda \geq 1-\alpha$, then

$$\int_0^{1-\alpha}\frac{|t-\alpha\lambda|}{(A_t)^2}dt=\int_0^{1-\alpha}\frac{-(t-\alpha\lambda)}{(A_t)^2}dt=m_3(\lambda,\alpha,u,v). \qquad (2.7)$$

where

$$\begin{aligned}m_3(\lambda,\alpha,u,v) &:= \frac{u(-1+\alpha\lambda)-v\alpha\lambda+(v(-1+\alpha)-u\alpha)\log(v(-1+\alpha)-u\alpha)}{(u-v)^2(v+u\alpha-v\alpha)}\\ &-\frac{u(-1+\alpha\lambda)-v\alpha\lambda-u\log(-u)}{u(u-v)^2}\end{aligned}$$

(b) (i) If $1+\lambda(\alpha-1) \leq 1-\alpha$, then

$$\int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}dt=\int_{1-\alpha}^{1}\frac{t-(1+\lambda(\alpha-1))}{(A_t)^2}dt=m_4(\lambda,\alpha,u,v). \qquad (2.8)$$

where

$$\begin{aligned}m_4(\lambda,\alpha,u,v) &:= \frac{v+u\lambda-v\lambda-u\alpha\lambda+v\alpha\lambda+v\log(-v)}{(u-v)^2v}\\ &-\frac{v+u\lambda-v\lambda-u\alpha\lambda+v\alpha\lambda+(v+u\alpha-v\alpha)\log(v(-1+\alpha)-u\alpha)}{(u-v)^2(u+v\alpha-v\alpha)}\end{aligned}$$

(*ii*) If $1+\lambda(\alpha-1)\geq 1-\alpha$, then

$$
\begin{aligned}
&\int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}dt\\
=&\int_{1-\alpha}^{1+\lambda(\alpha-1)}\frac{-(t-(1+\lambda(\alpha-1)))}{(A_t)^2}dt+\int_{1+\lambda(\alpha-1)}^{1}\frac{t-(1+\lambda(\alpha-1))}{(A_t)^2}dt\\
=&\ m_5(\lambda,\alpha,u,v)+m_6(\lambda,\alpha,u,v). \qquad (2.9)
\end{aligned}
$$

where

$$
\begin{aligned}
m_5(\lambda,\alpha,u,v) &:= -\frac{u(-1+\alpha)\lambda+v(-1+\lambda-\alpha\lambda)+(v(-1+\alpha)-u\alpha)\log(v(-1+\alpha)-u\alpha)}{(u-v)^2(v+u\alpha-v\alpha)}\\
&\quad -\frac{1+\log(u(-1+\alpha)\lambda+v(-1+\lambda-\alpha\lambda))}{(u-v)^2}\\
m_6(\lambda,\alpha,u,v) &:= \frac{u\lambda-v\lambda-u\alpha\lambda+v\alpha\lambda+v\log(-v)-v\log(u(-1+\alpha)\lambda+v(-1+\lambda-\alpha\lambda))}{(u-v)^2v}
\end{aligned}
$$

Since $|f'|^\mu$ for $\mu>1$, where $|f'|^\mu$ be harmonically convex on the interval $[u,v]$, as $t\in[0,1]$

$$\left|f'\left(\frac{uv}{tv+(1-t)u}\right)\right|^\mu\leq t|f'(u)|^\mu+(1-t)|f'(v)|^\mu.$$

hence, by calculation, we get

(c) (*i*) If $\alpha\lambda\leq 1-\alpha$, then

$$
\begin{aligned}
&\int_0^{1-\alpha}\frac{|t-\alpha\lambda|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|^\mu dt\\
\leq&\int_0^{\alpha\lambda}\frac{-(t-\alpha\lambda)}{(A_t)^2}[t|f'(u)|^\mu+(1-t)|f'(v)|^\mu]dt\\
&+\int_{\alpha\lambda}^{1-\alpha}\frac{t-\alpha\lambda}{(A_t)^2}[t|f'(u)|^\mu+(1-t)|f'(v)|^\mu]dt\\
=&\left[\int_0^{\alpha\lambda}\frac{-(t-\alpha\lambda)}{(A_t)^2}(t)dt+\int_{\alpha\lambda}^{1-\alpha}\frac{t-\alpha\lambda}{(A_t)^2}(t)dt\right]|f'(u)|^\mu\\
&+\left[\int_0^{\alpha\lambda}\frac{-(t-\alpha\lambda)}{(A_t)^2}(1-t)dt+\int_{\alpha\lambda}^{1-\alpha}\frac{t-\alpha\lambda}{(A_t)^2}(1-t)dt\right]|f'(v)|^\mu\\
=&\ [m_7(\lambda,\alpha,u,v)+m_8(\lambda,\alpha,u,v)]|f'(u)|^\mu\\
&+[m_9(\lambda,\alpha,u,v)+m_{10}(\lambda,\alpha,u,v)]|f'(v)|^\mu. \qquad (2.10)
\end{aligned}
$$

where

$$
\begin{aligned}
m_7(\lambda,\alpha,u,v) &:= \frac{2(-u+v)\alpha\lambda-(u(-2+\alpha\lambda)-v\alpha\lambda)\log\left(\frac{-u}{u(-1+\alpha\lambda)-v\alpha\lambda}\right)}{(u-v)^3}\\
m_8(\lambda,\alpha,u,v) &:= \frac{v+u\alpha-v\alpha-\frac{u(u-u\alpha\lambda+v\alpha\lambda)}{v+u\alpha-v\alpha}+(u(-2+\alpha\lambda)-v\alpha\lambda)\log(v(-1+\alpha)-u\alpha)}{(v-u)^3}\\
&\quad +\frac{(u-v)\alpha\lambda+(-u(-2+\alpha\lambda)+v\alpha\lambda)\log(u(-1+\alpha\lambda)-v\alpha\lambda)}{(v-u)^3}\\
m_9(\lambda,\alpha,u,v) &:= \frac{(v-u)(u+v\alpha\lambda)+u(u+v-u\alpha\lambda+v\alpha\lambda)\log(-u)}{u(v-u)^3}\\
&\quad -\frac{((u-v)(-1+\alpha\lambda)+(u+v-u\alpha\lambda+v\alpha\lambda)\log(u(-1+\alpha\lambda)-v\alpha\lambda))}{(v-u)^3}
\end{aligned}
$$

$$\begin{aligned} m_{10}(\lambda,\alpha,u,v) \;:=\; & -\frac{(u-v)(u\alpha^2+u(-1+2\alpha-\alpha^2+\alpha\lambda))}{(u-v)^3(v(-1+\alpha)-u\alpha)} \\ & +\frac{(v(-1+\alpha)-u\alpha)(u+v-u\alpha\lambda+v\alpha\lambda)\log(v+u\alpha-v\alpha)}{(u-v)^3(v(-1+\alpha)-u\alpha)} \\ & -\frac{-(u-v)(-1+\alpha\lambda)+(u(-1+\alpha\lambda)-v(1+\alpha\lambda))\log(u-u\alpha\lambda+v\alpha\lambda)}{(u-v)^3} \end{aligned}$$

(ii) If $\alpha\lambda \geq 1-\alpha$, then

$$\begin{aligned} & \int_0^{1-\alpha} \frac{|t-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|^{\mu} dt \\ \leq\; & \int_0^{1-\alpha} \frac{-(t-\alpha\lambda)}{(A_t)^2}[t|f'(u)|^{\mu}+(1-t)|f'(v)|^{\mu}]dt \\ =\; & \int_0^{1-\alpha} \frac{-(t-\alpha\lambda)}{(A_t)^2}(t)dt|f'(u)|^{\mu}+\int_0^{1-\alpha} \frac{-(t-\alpha\lambda)}{(A_t)^2}(1-t)dt|f'(v)|^{\mu} \\ =\; & m_{11}(\lambda,\alpha,u,v,)|f'(u)|^{\mu}+m_{12}(\lambda,\alpha,u,v)|f'(v)|^{\mu}. \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} m_{11}(\lambda,\alpha,u,v) := & \frac{v+u\alpha-v\alpha+\frac{u(u(-1+\alpha\lambda)-v\alpha\lambda)}{v+u\alpha-v\alpha}+(u(-2+\alpha\lambda)-v\alpha\lambda)\log(v(-1+\alpha)-u\alpha)}{(u-v)^3} \\ & -\frac{((u-v)\alpha\lambda+(u(-2+\alpha\lambda)-v\alpha\lambda)\log(-u))}{(u-v)^3} \\ m_{12}(\lambda,\alpha,u,v) := & \frac{v+u\alpha-v\alpha-\frac{v(u-u\alpha\lambda+v\alpha\lambda)}{v+u\alpha-v\alpha}+(u(-1+\alpha\lambda)-v(1+\alpha)\lambda)\log(v(-1+\alpha)-u\alpha)}{(v-u)^3} \\ & +\frac{-(u-v)(u+v\alpha\lambda)+u(u+v-u\alpha\lambda+u\alpha\lambda)\log(-u)}{u(v-u)^3} \end{aligned}$$

(d) (i) If $1+\lambda(\alpha-1) \leq 1-\alpha$, then

$$\begin{aligned} & \int_{1-\alpha}^{1} \frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|^{\mu} dt \\ \leq\; & \int_{1-\alpha}^{1} \frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}[t|f'(u)|^{\mu}+(1-t)|f'(v)|^{\mu}]dt \\ =\; & \int_{1-\alpha}^{1} \frac{t-(1+\lambda(\alpha-1))}{(A_t)^2}(t)dt|f'(u)|^{\mu} \\ & +\int_{1-\alpha}^{1} \frac{t-(1+\lambda(\alpha-1))}{(A_t)^2}(1-t)dt|f'(v)|^{\mu} \\ =\; & m_{13}(\lambda,\alpha,u,v)|f'(u)|^{\mu}+m_{14}(\lambda,\alpha,u,v)|f'(v)|^{\mu}. \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} m_{13}(\lambda,\alpha,u,v) := & \frac{-(u-v)(-v+u(-1+\alpha)\lambda)+v(u+v+u\lambda-v\lambda-u\alpha\lambda+v\alpha\lambda)\log(-v)}{(u-v)^3 v} \\ & +\frac{((u-v)(-v(-1+\alpha)^2+u(\alpha^2-\lambda+\alpha\lambda)))}{(v+u\alpha-v\alpha)(u-v)^3} \\ & +\frac{(u(-1+(-1+\alpha)\lambda)+v(-1+\lambda-\alpha\lambda))\log[v(-1+\alpha)-u\alpha]}{(u-v)^3} \\ m_{14}(\lambda,\alpha,u,v) := & -\frac{-(u-v)(-1+\alpha)\lambda+(-u(-1+\alpha)\lambda+v(2+(-1+\alpha)\lambda))\log(-v)}{(u-v)^3} \\ & +\frac{-v-u\alpha+v\alpha+\frac{v(-u(-1+\alpha)\lambda+v(1+(-1+\alpha)\lambda))}{v+u\alpha-v\alpha}}{(u-v)^3} \end{aligned}$$

$$+\frac{-u(-1+\alpha)\lambda+v(2+(-1+\alpha)\lambda)\log(v(-1+\alpha)-u\alpha)}{(u-v)^3}$$

(ii) If $1+\lambda(\alpha-1)\geq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|^{\mu}dt\\
\leq\ &\int_{1-\alpha}^{1+\lambda(\alpha-1)}\frac{-(t-(1+\lambda(\alpha-1)))}{(A_t)^2}[t|f'(u)|^{\mu}+(1-t)|f'(v)|^{\mu}]dt\\
&+\int_{1+\lambda(\alpha-1)}^{1}\frac{t-(1+\lambda(\alpha-1))}{(A_t)^2}[t|f'(u)|^{\mu}+(1-t)|f'(v)|^{\mu}]dt\\
=\ &\left[\int_{1-\alpha}^{1+\lambda(\alpha-1)}\frac{-(t-(1+\lambda(\alpha-1)))}{(A_t)^2}(t)dt\right.\\
&\left.+\int_{1+\lambda(\alpha-1)}^{1}\frac{t-(1+\lambda(\alpha-1))}{(A_t)}(t)dt\right]|f'(u)|^{\mu}\\
&+\left[\int_{1-\alpha}^{1+\lambda(\alpha-1)}\frac{-(t-(1+\lambda(\alpha-1)))}{(A_t)^2}(1-t)dt\right.\\
&\left.+\int_{1+\lambda(\alpha-1)}^{1}\frac{t-(1+\lambda(\alpha-1))}{(A_t)^2}(1-t)dt\right]|f'(v)|^{\mu}\\
=\ &[m_{15}(\lambda,\alpha,u,v)+m_{16}(\lambda,\alpha,u,v)]|f'(u)|^{\mu}\\
&+[m_{17}(\lambda,\alpha,u,v)+m_{18}(\lambda,\alpha,u,v)]|f'(v)|^{\mu}. \qquad (2.13)
\end{aligned}$$

where

$$\begin{aligned}
m_{15}(\lambda,\alpha,u,v) :=\ &-\frac{v+u\alpha-v\alpha+\frac{u(u(-1+\alpha)\lambda+v(-1+\lambda-\alpha\lambda))}{v+u\alpha-v\alpha}}{(u-v)^3}\\
&+\frac{(u(-1+(-1+\alpha)\lambda)+v(-1+\lambda-\alpha\lambda))}{(u-v)^3}\\
&\times\log\left(\frac{u(-1+\alpha)\lambda+v(-1+\lambda-\alpha\lambda)}{v(-1+\alpha)-u\alpha}\right)\\
&-\frac{-u+v+u\lambda-v\lambda-u\alpha\lambda+v\alpha\lambda}{(v-u)^3}\\
m_{16}(\lambda,\alpha,u,v) :=\ &\frac{(u-v)(v-u(-1+\alpha)\lambda)+v(u\lambda-v\lambda-u\alpha\lambda+u\alpha\lambda+u+v)\log(-v)}{(u-v)^3v}\\
&+\frac{(u(-1+(-1+\alpha)\lambda)+v(-1+\lambda-\alpha\lambda))\log(u(-1+\alpha)\lambda+v(-1+\lambda-\alpha\lambda))}{(u-v)^3}\\
&-\frac{(1+(-1+\alpha)\lambda)}{(u-v)^2}\\
m_{17}(\lambda,\alpha,u,v) :=\ &-\frac{-v-u\alpha+v\alpha+\frac{v(-u(-1+\alpha)\lambda+v(1+(-1+\alpha)\lambda))}{v+u\alpha-v\alpha}}{(u-v)^3}\\
&+\frac{-u\lambda+v\lambda+u\alpha\lambda-v\alpha\lambda}{(u-v)^3}+\frac{(-u(-1+\alpha)\lambda+v(2+(-1+\alpha)\lambda))}{(u-v)^3}\\
&\times\log\left(\frac{u(-1+\alpha)\lambda+v(-1+\lambda-\alpha\lambda)}{v(-1+\alpha)-u\alpha}\right)\\
m_{18}(\lambda,\alpha,u,v) :=\ &-\frac{(v-u)(-1+\alpha)\lambda+(v(2+(-1+\alpha)\lambda)-u(-1+\alpha)\lambda)\log(-v)}{(u-v)^3}\\
&+\frac{(-u(-1+\alpha)\lambda+v(2+(-1+\alpha)\lambda))\log(u(-1+\alpha)\lambda+v(-1+\lambda-\alpha\lambda))}{(u-v)^3}
\end{aligned}$$

$$+\frac{-u\lambda+v\lambda+u\alpha\lambda-v\alpha\lambda}{(u-v)^3}$$

By substituting (2.6) to (2.13) in equation (2.5) gives the required result. $\square$

Corollary 1 *[30] Assuming that $f: M=[u,v]\subseteq \mathbb{R}_+=(0,\infty)\longrightarrow \mathbb{R}$ is differentiable on the interior, $M^o=(u,v)$ of M where $f'\in L[u,v]$ for u,v in M with $u<v$. If $|f'|^\mu$ is harmonically convex on M for $\mu\in(1,\infty)$ with $\frac{1}{\gamma}+\frac{1}{\mu}=1$, then*

$$\left|\frac{uv}{b-a}\int_u^v \frac{f(z)}{z^2}dz-f\left(\frac{2uv}{u+v}\right)\right|\leq uv(v-u)[w_5^{\frac{1}{\gamma}}(\lambda,u,v)\{w_1(\lambda,u,v)|f'(u)|^\mu+w_2(\lambda,u,v)|f'(v)|^\mu\}^{\frac{1}{\mu}}$$
$$+w_6^{\frac{1}{\gamma}}(\lambda,u,v)\{w_3(\lambda,u,v)|f'(u)|^\mu+w_4(\lambda,u,v)|f'(v)|^\mu\}^{\frac{1}{\mu}}.$$

where

$$\begin{aligned}
w_1(\lambda,u,v) &:= -\frac{-3u^2+2uv+v^2+u(u+v)\log(16)}{2(u-v)^3(u+v)}-\frac{2u\log(\frac{u}{u+v})}{(u-v)^3}\\
w_2(\lambda,u,v) &:= -\frac{(u-v-(u+v)\log(-u))}{(u-v)^3}+\frac{(u-v)^2+2(u+v)^2\log(\frac{2}{-u-v})}{2(u-v)^3(u+v)}\\
w_3(\lambda,u,v) &:= \frac{-3u^2+2uv+v^2-(u+v)^2\log(4)+2(u+v)^2\log(\frac{u+v}{v})}{2(u-v)^3(u+v)}\\
w_4(\lambda,u,v) &:= \frac{u^2+v^2(-3+\log(16))+uv(2+\log(16))}{2(u-v)^3(u+v)}+\frac{2v\log(\frac{v}{u+v})}{(u-v)^3}\\
w_5(\lambda,u,v) &:= \frac{u-v-(u+v)\log(2u)+(u+v)\log(u+v)}{(u-v)^2(u+v)}\\
w_6(\lambda,u,v) &:= \frac{-u+v+(u+v)(\log(-u-v)-\log(-2v))}{(u-v)^2(u+v)}
\end{aligned}$$

Proof. From (2.4), we have

$$\left|\frac{uv}{v-u}\int_u^v \frac{f(z)}{z^2}dz-f\left(\frac{2uv}{u+v}\right)\right|\leq uv(u-v)\left[\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|dt+\int_{\frac{1}{2}}^1\frac{|t-1|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|dt\right].$$

By power mean integral inequality, we have

$$\begin{aligned}
&\leq\ ab(b-a)\left[\left(\int_0^{\frac{1}{2}}\frac{|t|}{(A_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\frac{|t|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|^\mu dt\right)^{\frac{1}{\mu}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(A_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|^\mu dt\right)^{\frac{1}{\mu}}\right].
\end{aligned}$$

Since $|f'|^\mu$ for $\mu>1$, where $|f'|^\mu$ be harmonically convex on the interval $[u,v]$, as $t\in[0,1]$

$$\left|f'\left(\frac{uv}{tv+(1-t)u}\right)\right|^\mu\leq t|f'(u)|^\mu+(1-t)|f'(v)|^\mu$$

$$\begin{aligned}
&\left|\frac{uv}{v-u}\int_u^v\frac{f(z)}{z^2}dz-f\left(\frac{2uv}{u+v}\right)\right|\\
\leq\ & uv(v-u)\left[\left(\int_0^{\frac{1}{2}}\frac{|t|}{(A_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\frac{|t|}{(A_t)^2}[t|f'(u)|^\mu+(1-t)|f'(v)|^\mu]dt\right)^{\frac{1}{\mu}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(A_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(A_t)^2}[t|f'(u)|^\mu+(1-t)|f'(v)|^\mu]dt\right)^{\frac{1}{\mu}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= uv(v-u)\left[\left(\int_0^{\frac{1}{2}} \frac{t}{(A_t)^2}dt\right)^{\frac{1}{\gamma}} \left(\int_0^{\frac{1}{2}} \frac{t^2}{(A_t)^2}dt|f'(u)|^\mu + \int_0^{\frac{1}{2}} \frac{t(1-t)}{(A_t)^2}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}\right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 \frac{-(t-1)}{(A_t)^2}dt\right)^{\frac{1}{\gamma}} \left(\int_{\frac{1}{2}}^1 \frac{-(t-1)t}{(A_t)^2}dt|f'(u)|^\mu + \int_{\frac{1}{2}}^1 \frac{-(t-1)(1-t)}{(A_t)^2}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}\right] \\
&\le uv(v-u)[w_5^{\frac{1}{\gamma}}(\lambda,u,v)\{w_1(\lambda,u,v)|f'(u)|^\mu + w_2(\lambda,u,v)|f'(v)|^\mu\}^{\frac{1}{\mu}} \\
&\quad + w_6^{\frac{1}{\gamma}}(\lambda,u,v)\{w_3(\lambda,u,v)|f'(u)|^\mu + w_4(\lambda,u,v)|f'(v)|^\mu\}^{\frac{1}{\mu}}].
\end{aligned}
$$

□

Theorem 2 *[30] Assuming that $f : M = [u,v] \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L[u,v]$ for u,v in M with $u < v$ and $0 \le \alpha,\lambda \le 1$. If $|f'|^\mu$ is harmonically convex on M for $\mu \in (1,\infty)$, we have*

(a) *If $\alpha\lambda \le 1-\alpha \le 1+\lambda(\alpha-1)$, then*

$$
\begin{aligned}
|\Psi_f(\lambda,\alpha,u,v)| \le uv(v-u)&\left[(m_{19}(\lambda,\alpha,u,v,\gamma)+m_{20}(\lambda,\alpha,u,v,\gamma))^{\frac{1}{\gamma}}\left((1-\alpha)\frac{\left\{\left|f'\left(\frac{uv}{A_{1-\alpha}}\right)\right|^\mu + |f'(v)|^\mu\right\}}{2}\right)^{\frac{1}{\mu}}\right. \\
&\left. +(m_{23}(\lambda,\alpha,u,v,\gamma)+m_{24}(\lambda,\alpha,u,v,\gamma))^{\frac{1}{\gamma}}\left(\alpha\frac{\left\{|f'(u)|^\mu + \left|f'\left(\frac{uv}{A_{1-\alpha}}\right)\right|^\mu\right\}}{2}\right)^{\frac{1}{\mu}}\right].
\end{aligned}
$$

(b) *If $\alpha\lambda \le 1+\lambda(\alpha-1) \le 1-\alpha$, then*

$$
\begin{aligned}
|\Psi_f(\lambda,\alpha,u,v)| \le uv(v-u)&\left[(m_{19}(\lambda,\alpha,u,v,\gamma)+m_{20}(\lambda,\alpha,u,v,\gamma))^{\frac{1}{\gamma}}\left((1-\alpha)\frac{\left\{\left|f'\left(\frac{uv}{A_{1-\alpha}}\right)\right|^\mu + |f'(v)|^\mu\right\}}{2}\right)^{\frac{1}{\mu}}\right. \\
&\left. +(m_{22}(\lambda,\alpha,u,v,\gamma))^{\frac{1}{\gamma}}\left(\alpha\frac{\left\{|f'(u)|^\mu + \left|f'\left(\frac{uv}{A_{1-\alpha}}\right)\right|^\mu\right\}}{2}\right)^{\frac{1}{\mu}}\right].
\end{aligned}
$$

(c) *If $1-\alpha \le \alpha\lambda \le 1+\lambda(\alpha-1)$, then*

$$
\begin{aligned}
|\Psi_f(\lambda,\alpha,u,v)| \le uv(v-u)&\left[(m_{21}(\lambda,\alpha,u,v,\gamma))^{\frac{1}{\gamma}}\left((1-\alpha)\frac{\left\{\left|f'\left(\frac{uv}{A_{1-\alpha}}\right)\right|^\mu + |f'(v)|^\mu\right\}}{2}\right)^{\frac{1}{\mu}}\right. \\
&\left. +(m_{23}(\lambda,\alpha,u,v,\gamma)+m_{24}(\lambda,\alpha,u,v,\gamma))^{\frac{1}{\gamma}}\left(\alpha\frac{\left\{|f'(u)|^\mu + \left|f'\left(\frac{uv}{A_{1-\alpha}}\right)\right|^\mu\right\}}{2}\right)^{\frac{1}{\mu}}\right].
\end{aligned}
$$

Proof. By using Lemma 1 and Hölder's integral inequality, we get

$$
\begin{aligned}
|\Psi_f(\lambda,\alpha,u,v)| &\le uv(v-u)\left[\int_0^{1-\alpha} \frac{|t-\alpha\lambda|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|dt + \int_{1-\alpha}^1 \frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|dt\right] \\
&\le uv(v-u)\left(\int_0^{1-\alpha} \frac{|t-\alpha\lambda|^\gamma}{(A_t)^{2\gamma}}dt\right)^{\frac{1}{\gamma}}\left(\int_0^{1-\alpha}\left|f'\left(\frac{uv}{A_t}\right)\right|^\mu dt\right)^{\frac{1}{\mu}} \\
&\quad + \left(\int_{1-\alpha}^1 \frac{|t-(1+\lambda(\alpha-1))|^\gamma}{(A_t)^{2\gamma}}dt\right)^{\frac{1}{\gamma}}\left(\int_{1-\alpha}^1\left|f'\left(\frac{uv}{A_t}\right)\right|^\mu dt\right)^{\frac{1}{\mu}} \qquad (2.14)
\end{aligned}
$$

(a) (*i*) If $\alpha\lambda \leq 1-\alpha$, then

$$\begin{aligned}
&\int_0^{1-\alpha} \frac{|t-\alpha\lambda|^\gamma}{(A_t)^{2\gamma}} dt \\
= &\int_0^{\alpha\lambda} \frac{(-t+\alpha\lambda)^\gamma}{(A_t)^{2\gamma}} dt + \int_{\alpha\lambda}^{1-\alpha} \frac{(t-\alpha\lambda)^\gamma}{(A_t)^{2\gamma}} dt \\
= &m_{19}(\lambda,\alpha,u,v,\gamma) + m_{20}(\lambda,\alpha,u,v,\gamma). \qquad (2.15)
\end{aligned}$$

where

$$m_{19}(\lambda,\alpha,u,v,\gamma) := \frac{u^{-2\gamma}\alpha\lambda^{1+\gamma}{}_2F_1[1,2\gamma,2+\gamma,\alpha\lambda-\frac{v\alpha\lambda}{u}]}{(1+\gamma)} +$$

$$m_{20}(\lambda,\alpha,u,v,\gamma) := \frac{(v+u\alpha-v\alpha)^{1-2\gamma}(1-\alpha-\alpha\lambda)^{1+\gamma}\Gamma(1+\gamma){}_2\tilde{F}_1[1,2-\gamma,2+\gamma,\frac{(u-v)(-1+\alpha+\alpha\lambda)}{u(-1+\alpha\lambda)-v\alpha\lambda}]}{u-u\alpha\lambda+v\alpha\lambda}$$

(*ii*) If $\alpha\lambda \geq 1-\alpha$, then

$$\int_0^{1-\alpha} \frac{|t-\alpha\lambda|^\gamma}{(A_t)^{2\gamma}} dt = \int_0^{1-\alpha} \frac{(-t+\alpha\lambda)^\gamma}{(A_t)^{2\gamma}} dt = m_{21}(\lambda,\alpha,u,v,\gamma). \qquad (2.16)$$

where

$$\begin{aligned}
m_{21}(\lambda,\alpha,u,v,\gamma) := &-\frac{u^{1-2\gamma}\alpha\lambda^{1+\gamma}{}_2F_1[1,2-\gamma,2+\gamma,\frac{(u-v)\alpha\lambda}{u(-1+\alpha\lambda)-v\alpha\lambda}]}{(1+\gamma)(u(-1+\alpha)\lambda)-v\alpha\lambda} \\
&+\frac{(v+u\alpha-v\alpha)^{1-2\gamma}(-1+\alpha+\alpha\lambda)^{1+\gamma}{}_2F_1[1,2-\gamma,2+\gamma,\frac{(u-v)(-1+\alpha+\alpha\lambda)}{u(-1+\alpha\lambda)-v\alpha\lambda}]}{(1+\gamma)(u(-1+\alpha\lambda)-v\alpha\lambda)}
\end{aligned}$$

(b) (*i*) If $1+\lambda(\alpha-1) \leq 1-\alpha$, then

$$\begin{aligned}
\int_{1-\alpha}^{1} \frac{|t-(1+\lambda(\alpha-1))|^\gamma}{(A_t)^{2\gamma}} dt &= \int_{1-\alpha}^{1} \frac{(t-(1+\lambda(\alpha-1)))^\gamma}{(A_t)^{2\gamma}} dt \\
&= m_{22}(\lambda,\alpha,u,v,\gamma). \qquad (2.17)
\end{aligned}$$

where

$$\begin{aligned}
m_{22}(\lambda,\alpha,u,v,\gamma) := &-\frac{v^{1-2\gamma}(-1+\alpha)\lambda(\lambda-\alpha\ \lambda)^\gamma{}_2F_1[1,2-\gamma,2+\gamma,\frac{(u-v)(-1+\alpha)\lambda}{u(-1+\alpha)\lambda+v(-1+\lambda-\alpha\lambda)}]}{(1+\gamma)(-u(-1+\alpha)\lambda)+v(1+(-1+\alpha)\lambda)} \\
&+\frac{(v+u\alpha-v\alpha)^{1-2\gamma}(\lambda-\alpha(1+\lambda))^{1+\gamma}{}_2F_1[1,2-\gamma,2+\gamma,\frac{(u-v)(\alpha-\lambda+\alpha\lambda)}{u(-1+\alpha)\lambda+v(-1+\lambda-\alpha\lambda)}]}{(1+\gamma)(-u(-1+\alpha)\lambda)+v(1+(-1+\alpha)\lambda)}
\end{aligned}$$

(*ii*) If $1+\lambda(\alpha-1) \geq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^{1} \frac{|t-(1+\lambda(\alpha-1))|^\gamma}{(A_t)^{2\gamma}} dt \\
= &\int_{1-\alpha}^{1+\lambda(\alpha-1)} \frac{(-(t-(1+\lambda(\alpha-1))))^\gamma}{(A_t)^{2\gamma}} dt \\
&+\int_{1+\lambda(\alpha-1)}^{1} \frac{(t-(1+\lambda(\alpha-1)))^\gamma}{(A_t)^{2\gamma}} dt \\
= &m_{23}(\lambda,\alpha,u,v,\gamma) + m_{24}(\lambda,\alpha,u,v,\gamma). \qquad (2.18)
\end{aligned}$$

where

$$m_{23}(\lambda,\alpha,u,v,\gamma) := \frac{(v+u\alpha-v\alpha)^{1-2\gamma}(\alpha-\lambda+\alpha\lambda)^{1+\gamma}{}_2F_1[1,2-\gamma,2+\gamma,\frac{(u-v)(\alpha-\lambda+\alpha\lambda)}{-v+(u-v)(-1+\alpha)\lambda}]}{(1+\gamma)(v+(-u+v)(-1+\alpha)\lambda)}$$

$$m_{24}(\lambda,\alpha,u,v,\gamma) := -\frac{v^{1-2\gamma}(-1+\alpha)\lambda(\lambda-\alpha\lambda)^{\gamma}{}_2F_1[1,2-\gamma,2+\gamma,1-\frac{v}{v+(-u+v)(-1+\alpha)\lambda}]}{(1+\gamma)(v+(-u+v)(-1+\alpha)\lambda)}$$

(c) Consider,

$$\int_0^{1-\alpha}\left|f'\left(\frac{uv}{A_t}\right)\right|^{\mu}dt \tag{2.19}$$

Setting $x=\frac{uv}{A_t}$, so that $dt=\frac{-uv}{x^2(v-u)}dx$.

For $0\le t\le 1-\alpha$, we have $v\le x\le \frac{uv}{A_{1-\alpha}}$ and hence (2.19) becomes

$$\begin{aligned}\int_0^{1-\alpha}\left|f'\left(\frac{uv}{A_t}\right)\right|^{\mu}dt &= -\frac{uv}{(v-u)}\int_v^{\frac{uv}{A_{1-\alpha}}}\frac{|f'(x)|^{\mu}}{x^2}dx\\ &= \frac{uv}{(v-u)}\int_{\frac{uv}{A_{1-\alpha}}}^{v}\frac{|f'(x)|^{\mu}}{x^2}dx\\ &= \frac{uv}{(v-u)}\left(\frac{v\frac{uv}{A_{1-\alpha}}}{v-\frac{uv}{A_{1-\alpha}}}\right)\left(\frac{v-\frac{uv}{A_{1-\alpha}}}{v\frac{uv}{A_{1-\alpha}}}\right)\int_{\frac{uv}{A_{1-\alpha}}}^{v}\frac{|f'(x)|^{\mu}}{x^2}dx\end{aligned}$$

Using Hermite-Hadamard's inequality for relative harmonically convex functions, we have

$$\begin{aligned}\int_0^{1-\alpha}\left|f'\left(\frac{uv}{A_t}\right)\right|^{\mu}dt &\le \frac{uv}{(v-u)}\left(\frac{v-\frac{uv}{A_{1-\alpha}}}{v\frac{uv}{A_{1-\alpha}}}\right)\frac{[|f'(\frac{uv}{A_{1-\alpha}})|^{\mu}+|f'(v)|^{\mu}]}{2}\\ &= \frac{A_{1-\alpha}-u}{(v-u)}\frac{[|f'(\frac{uv}{A_{1-\alpha}})|^{\mu}+|f'(v)|^{\mu}]}{2}\\ &= \frac{\alpha u+(1-\alpha)v-u}{(v-u)}\frac{[|f'(\frac{uv}{A_{1-\alpha}})|^{\mu}+|f'(v)|^{\mu}]}{2}\\ &\le (1-\alpha)\frac{[|f'(\frac{uv}{A_{1-\alpha}})|^{\mu}+|f'(v)|^{\mu}]}{2}\end{aligned} \tag{2.20}$$

Above Inequality holds for $\alpha=1$.

(d) Consider,

$$\int_{1-\alpha}^{1}\left|f'\left(\frac{uv}{A_t}\right)\right|^{\mu}dt \tag{2.21}$$

Setting $x=\frac{uv}{A_t}$, so that $dt=\frac{-uv}{x^2(v-u)}dx$.

For $1-\alpha\le t\le 1$ we have $\frac{uv}{A_{1-\alpha}}\le x\le u$ and hence (2.21) becomes

$$\begin{aligned}\int_{1-\alpha}^{1}\left|f'\left(\frac{uv}{A_t}\right)\right|^{\mu}dt &= -\frac{uv}{(v-u)}\int_{\frac{uv}{A_{1-\alpha}}}^{u}\frac{|f'(x)|^{\mu}}{x^2}dx\\ &= \frac{uv}{(v-u)}\int_u^{\frac{uv}{A_{1-\alpha}}}\frac{|f'(x)|^{\mu}}{x^2}dx\\ &= \frac{uv}{(v-u)}\left(\frac{\frac{\{u^2+(v-u)\}}{A_{1-\alpha}}}{\frac{uv}{A_{1-\alpha}}-u}\right)\left(\frac{\frac{uv}{A_{1-\alpha}}-u}{\frac{u^2v}{A_{1-\alpha}}}\right)\int_u^{\frac{uv}{A_{1-\alpha}}}\frac{|f'(x)|^{\mu}}{x^2}dx\end{aligned}$$

Using Hermite-Hadamard's inequality for harmonically convex functions, we have

$$\begin{aligned}\int_{1-\alpha}^{1}\left|f'\left(\frac{uv}{A_t}\right)\right|^{\mu}dt &\le \frac{uv}{(v-u)}\left(\frac{\frac{uv}{A_{1-\alpha}}-u}{\frac{u^2v}{A_{1-\alpha}}}\right)\frac{[|f'(u)|^{\mu}+|f'(\frac{uv}{A_{1-\alpha}})|^{\mu}]}{2}\\ &= \frac{v-A_{1-\alpha}}{(v-u)}\frac{[|f'(u)|^{\mu}+|f'(\frac{uv}{A_{1-\alpha}})|^{\mu}]}{2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{v-\alpha u-(1-\alpha)(v)}{(v-u)}\frac{[|f'(u)|^{\mu}+|f'(\frac{uv}{A_{1-\alpha}})|^{\mu}]}{2}\\
&\le \alpha\frac{[|f'(u)|^{\mu}+|f'(\frac{uv}{A_{1-\alpha}})|^{\mu}]}{2} \qquad (2.22)
\end{aligned}
$$

Above Inequality holds for $\alpha = 0$.

By substituting (2.15) to (2.18), (2.20), (2.22) in equation (2.14) gives the required result. □

Corollary 2 *[30] Assuming that $f : M = [u, v] \subseteq \mathbb{R}_+ = (0, \infty) \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L[u, v]$ for u, v in M with $u < v$. If $|f'|^{\mu}$ is harmonically convex on M for $\mu \in (1, \infty)$, then*

$$
\left|\frac{uv}{(v-u)}\int_u^v \frac{f(z)}{z^2}dz - f\left(\frac{2uv}{u+v}\right)\right| \le uv(v-u)\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{1-\frac{1}{\mu}}[\{w_7(\lambda,u,v,\mu)|f'(u)|^{\mu}+w_8(\lambda,u,v,\mu)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}+\{w_9(\lambda,u,v,\mu)|f'(u)|^{\mu}+w_{10}(\lambda,u,v,\mu)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].
$$

where

$$
\begin{aligned}
w_7(\lambda,u,v,\mu) &:= \frac{u^{2-2\mu}}{2(u-v)^2(1-3\mu+2\mu^2)} + \frac{2^{-3+2\mu}(u+v)^{1-2\mu}(v-2v\mu+u(-3+2\mu))}{(u-v)^2(1-3\mu+2\mu^2)}\\
w_8(\lambda,u,v,\mu) &:= \frac{2^{-3+2\mu}(v+u)^{1-2\mu}(v(3-2\mu)+u(-1+2\mu))}{(u-v)^2(1-3\mu+2\mu^2)} - \frac{u^{1-2\mu}(-2v(-1+\mu)+u(-1+2\mu))}{2(u-v)^2(1-3\mu+2\mu^2)}\\
w_9(\lambda,u,v,\mu) &:= \frac{v^{1-2\mu}(v+2u(-1+\mu)-2v\mu)}{2(u-v)^2(1-3\mu+2\mu^2)} - \frac{2^{-3+2\mu}(u+v)^{1-2\mu}(v-2v\mu+u(-3+2\mu))}{(u-v)^2(1-3\mu+2\mu^2)}\\
w_{10}(\lambda,u,v,\mu) &:= \frac{v^{2-2\mu}}{2(u-v)^2(1-3\mu+2\mu^2)} - \frac{2^{-3+2\mu}(u+v)^{1-2\mu}(v(3-2\mu)+u(-1+2\mu))}{(u-v)^2(1-3\mu+2\mu^2)}
\end{aligned}
$$

Proof. From (2.4), we have

$$
\left|\frac{uv}{(v-u)}\int_u^v \frac{f(z)}{z^2}dz - f\left(\frac{2uv}{u+v}\right)\right| \le uv(v-u)\left[\int_0^{\frac{1}{2}} \frac{|t|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|dt + \int_{\frac{1}{2}}^1 \frac{|(t-1)|}{(A_t)^2}\left|f'\left(\frac{uv}{A_t}\right)\right|dt\right].
$$

Since $|f'|^{\mu}$ for $\mu > 1$, where $|f'|^{\mu}$ be harmonically convex on the interval $[u, v]$, as $t \in [0, 1]$, then

$$
\left|f'\left(\frac{uv}{tv+(1-t)u}\right)\right|^{\mu} \le t|f'(u)|^{\mu} + (1-t)|f'(v)|^{\mu}
$$

Using Hölder integral inequality, we have

$$
\begin{aligned}
&\left|\frac{uv}{(v-u)}\int_u^v \frac{f(z)}{z^2}dz - f\left(\frac{2uv}{u+v}\right)\right|\\
\le\ & uv(v-u)\left(\int_0^{\frac{1}{2}} t^{\gamma}dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}} \frac{1}{(A_t)^{2\mu}}(t|f'(u)|^{\mu}+(1-t)|f'(v)|^{\mu})dt\right)^{\frac{1}{\mu}}\\
&+\left(\int_{\frac{1}{2}}^1 (1-t)^{\gamma}dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^1 \frac{1}{(A_t)^{2\mu}}(t|f'(u)|^{\mu}+(1-t)|f'(v)|^{\mu})dt\right)^{\frac{1}{\mu}}\\
\le\ & uv(v-u)\left(\int_0^{\frac{1}{2}} t^{\gamma}dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}} \frac{t}{(A_t)^{2\mu}}dt|f'(u)|^{\mu} + \int_0^{\frac{1}{2}} \frac{(1-t)}{(A_t)^{2\mu}}dt|f'(v)|^{\mu}\right)^{\frac{1}{\mu}}\\
&+\left(\int_{\frac{1}{2}}^1 (1-t)^{\gamma}dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^1 \frac{t}{(A_t)^{2\mu}}dt|f'(u)|^{\mu} + \int_{\frac{1}{2}}^1 \frac{(1-t)}{(A_t)^{2\mu}}dt|f'(v)|^{\mu}\right)^{\frac{1}{\mu}}\\
\le\ & uv(v-u)\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{1-\frac{1}{\mu}}[\{w_7(\lambda,u,v,\mu)|f'(u)|^{\mu} + w_8(\lambda,u,v,\mu)
\end{aligned}
$$

$$|f'(v)|^{\mu}\}^{\frac{1}{\mu}} + \{w_9(\lambda,u,v,\mu)|f'(u)|^{\mu} + w_{10}(\lambda,u,v,\mu)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].$$

□

Theorem 3 *[30] Assuming that $f : M = [u,v] \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L[u,v]$ for u,v in M with $u < v$ and $0 \leq \alpha,\lambda \leq 1$. If $|f'|$ is harmonically convex on M, we have*

(a) *If $\alpha\lambda \leq 1-\alpha \leq 1+\lambda(\alpha-1)$, then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,u,v)| \leq\ & uv(v-u)[\{(m_7(\lambda,\alpha,u,v)+m_8(\lambda,\alpha,u,v))+(m_{15}(\lambda,\alpha,u,v)\\ & +m_{16}(\lambda,\alpha,u,v))\}|f'(u)|+\{(m_9(\lambda,\alpha,u,v)+m_{10}(\lambda,\alpha,u,v))\\ & +(m_{17}(\lambda,\alpha,u,v)+m_{18}(\lambda,\alpha,u,v))\}|f'(v)|].\end{aligned}$$

(b) *If $\alpha\lambda \leq 1+\lambda(\alpha-1) \leq 1-\alpha$, then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,u,v)| \leq\ & uv(v-u)[\{(m_7(\lambda,\alpha,u,v)+m_8(\lambda,\alpha,u,v))+m_{13}(\lambda,\alpha,u,v)\}\\ & |f'(u)|+\{(m_9(\lambda,\alpha,u,v)+m_{10}(\lambda,\alpha,u,v))+m_{14}(\lambda,\alpha,u,v)\}|f'(v)|].\end{aligned}$$

(c) *If $1-\alpha \leq \alpha\lambda \leq 1+\lambda(\alpha-1)$, then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,u,v))| \leq\ & uv(v-u)[\{(m_{15}(\lambda,\alpha,u,v)+m_{16}(\lambda,\alpha,u,v))+m_{11}(\lambda,\alpha,u,v)\}\\ & |f'(u)|+\{(m_{17}(\lambda,\alpha,u,v)+m_{18}(\lambda,\alpha,u,v))+m_{12}(\lambda,\alpha,u,v)\}|f'(v)|].\end{aligned}$$

where the values of $m_7(\lambda,\alpha,u,v)$ to $m_{18}(\lambda,\alpha,u,v)$ are defined in Theorem 1.

Proof. By using Lemma 1, we have

$$|\Psi_f(\lambda,\alpha,u,v)| = uv(u-v)\left[\int_0^{1-\alpha}\frac{|t-\alpha\lambda|}{{A_t}^2}\left|f'\left(\frac{uv}{A_t}\right)\right|dt+\int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{{A_t}^2}\left|f'\left(\frac{uv}{A_t}\right)\right|dt\right]$$

Since $|f'|$ be harmonically convex on the interval $[u,v]$, as $t \in [0,1]$

$$\left|f'\left(\frac{uv}{tv+(1-t)u}\right)\right| \leq t|f'(u)|+(1-t)|f'(v)|$$

$$\begin{aligned}|\Psi_f(\lambda,\alpha,u,v)| \leq\ & uv(v-u)\left[\int_0^{1-\alpha}\frac{|t-\alpha\lambda|}{(A_t)^2}[t|f'(u)|+(1-t)|f'(v)|]dt\right.\\ & \left.+\int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}[t|f'(u)|+(1-t)|f'(v)|]dt\right]\\ \leq\ & uv(v-u)\left[\left\{\int_0^{1-\alpha}\frac{|t-\alpha\lambda|}{(A_t)^2}tdt|f'(u)|+\int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}tdt|f'(v)|\right\}\right.\\ & +\left\{\int_0^{1-\alpha}\frac{|t-\alpha\lambda|}{(A_t)^2}(1-t)dt|f'(u)|\right.\\ & \left.\left.+\int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}(1-t)dt|f'(v)|\right\}\right]\end{aligned} \tag{2.23}$$

(a) (i) If $\alpha\lambda \leq 1-\alpha$, then

$$\begin{aligned}
&\int_0^{1-\alpha} \frac{|t-\alpha\lambda|}{(A_t)^2} t dt \\
&= \int_0^{\alpha\lambda} \frac{-(t-\alpha\lambda)}{(A_t)^2}(t)dt + \int_{\alpha\lambda}^{1-\alpha} \frac{t-\alpha\lambda}{(A_t)^2}(t)dt \\
&= \int_0^{\alpha\lambda} \frac{-(t-\alpha\lambda)}{((1-t)u+tv)^2}(t)dt + \int_{\alpha\lambda}^{1-\alpha} \frac{t-\alpha\lambda}{((1-t)u+tv)^2}(t)dt \\
&= m_7(\lambda,\alpha,u,v) + m_8(\lambda,\alpha,u,v). \qquad (2.24)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{1-\alpha} \frac{|t-\alpha\lambda|}{(A_t)^2}(1-t) dt \\
&= \int_0^{\alpha\lambda} \frac{-(t-\alpha\lambda)}{(A_t)^2}(1-t)dt + \int_{\alpha\lambda}^{1-\alpha} \frac{t-\alpha\lambda}{(A_t)^2}(1-t)dt \\
&= \int_0^{\alpha\lambda} \frac{-(t-\alpha\lambda)}{((1-t)u+tv)^2}(1-t)dt + \int_{\alpha\lambda}^{1-\alpha} \frac{t-\alpha\lambda}{((1-t)u+tv)^2}(1-t)dt \\
&= m_9(\lambda,\alpha,u,v) + m_{10}(\lambda,\alpha,u,v). \qquad (2.25)
\end{aligned}$$

(ii) If $\alpha\lambda \geq 1-\alpha$, then

$$\begin{aligned}
\int_0^{1-\alpha} \frac{|t-\alpha\lambda|}{(A_t)^2}(t) dt &= \int_0^{1-\alpha} \frac{-(t-\alpha\lambda)}{(A_t)^2}(t)dt \\
&= \int_0^{1-\alpha} \frac{-(t-\alpha\lambda)}{((1-t)u+tv)^2}(t)dt \\
&= m_{11}(\lambda,\alpha,u,v). \qquad (2.26)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{1-\alpha} \frac{|t-\alpha\lambda|}{(A_t)^2}(1-t) dt &= \int_0^{1-\alpha} \frac{-(t-\alpha\lambda)}{(A_t)^2}(1-t)dt \\
&= \int_0^{1-\alpha} \frac{-(t-\alpha\lambda)}{((1-t)u+tv)^2}(1-t)dt \\
&= m_{12}(\lambda,\alpha,u,v). \qquad (2.27)
\end{aligned}$$

(b) (i) If $1+\lambda(\alpha-1) \leq 1-\alpha$, then

$$\begin{aligned}
\int_{1-\alpha}^{1} \frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}(t) dt &= \int_{1-\alpha}^{1} \frac{t-(1+\lambda(\alpha-1))}{(A_t)^2}(t)dt \\
&= \int_{1-\alpha}^{1} \frac{t-(1+\lambda(\alpha-1))}{((1-t)u+tv)^2}(t)dt \\
&= m_{13}(\lambda,\alpha,\beta,u,v). \qquad (2.28)
\end{aligned}$$

and

$$\begin{aligned}
\int_{1-\alpha}^{1} \frac{|t-(1+\lambda(1-\alpha))|}{(A_t)^2}(1-t) dt &= \int_{1-\alpha}^{1} \frac{t-(1+\lambda(\alpha-1))}{(A_t)^2}(1-t)dt \\
&= \int_{1-\alpha}^{1} \frac{t-(1+\lambda(\alpha-1))}{((1-t)u+tv)^2}(1-t)dt \\
&= m_{14}(\lambda,\alpha,u,v). \qquad (2.29)
\end{aligned}$$

(ii) If $1+\lambda(\alpha-1)\geq 1-\alpha$, then

$$\begin{aligned}
& \int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}h(t)dt \\
= & \int_{1-\alpha}^{1+\lambda(\alpha-1)}\frac{-(t-(1+\lambda(\alpha-1)))}{(A_t)^2}(t)dt \\
& +\int_{1+\lambda(\alpha-1)}^{1}\frac{t-(1+\lambda(\alpha-1))}{(A_t)^2}(t)dt \\
= & \int_{1-\alpha}^{1+\lambda(\alpha-1)}\frac{-(t-(1+\lambda(\alpha-1)))}{((1-t)u+tv)^2}(t)dt \\
& +\int_{1+\lambda(\alpha-1)}^{1}\frac{t-(1+\lambda(\alpha-1))}{((1-t)u+tv)^2}(t)dt \\
= & \; m_{15}(\lambda,\alpha,u,v)+m_{16}(\lambda,\alpha,u,v). \qquad (2.30)
\end{aligned}$$

and

$$\begin{aligned}
& \int_{1-\alpha}^{1}\frac{|t-(1+\lambda(\alpha-1))|}{(A_t)^2}(1-t)dt \\
= & \int_{1-\alpha}^{1+\lambda(\alpha-1)}\frac{-(t-(1+\lambda(\alpha-1)))}{(A_t)^2}(1-t)dt \\
& +\int_{1+\lambda(\alpha-1)}^{1}\frac{t-(1+\lambda(\alpha-1))}{(A_t)^2}(1-t)dt \\
= & \int_{1-\alpha}^{1+\lambda(\alpha-1)}\frac{-(t-(1+\lambda(\alpha-1)))}{((1-t)u+tv)^2}(1-t)dt \\
& +\int_{1+\lambda(\alpha-1)}^{1}\frac{t-(1+\lambda(\alpha-1))}{((1-t)u+tv)^2}(1-t)dt \\
= & \; m_{17}(\lambda,\alpha,u,v)+m_{18}(\lambda,\alpha,u,v). \qquad (2.31)
\end{aligned}$$

By substituting (2.24) to (2.31) in equation (2.23) gives the required result. □

HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR HARMONICALLY RELATIVE PREINVEX FUNCTIONS

CHAPTER 3

HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR HARMONICALLY RELATIVE PREINVEX FUNCTIONS

3.1 Introduction:

In the recent years, researchers have motivated and inspired to establish the theory of convex function in diverse field of applied and pure sciences. Weir and Mond [33] had given a significant generalization of convex functions by introducing preinvex functions.

Further, established forms have been made by researchers generalizing the harmonic preinvex functions, relative preinvex functions, relative harmonic preinvex functions, Breckner type of s-harmonic preinvex functions, Godunova-Levin type of s-harmonic preinvex functions and harmonic P-preinvex functions [14, 19, 23, 26, 27]. Our aim is to describe several new upper bounds of Hermite-Hadamard type integral inequalities for relative harmonically preinvex functions and their variant forms are available in the literature [4, 13, 16, 17, 20, 22, 25, 28].

Now, we recall literature review and background [2, 26, 28, 30, 34].

3.1.1 Relative Harmonic Preinvex Function

Let $h : [0, 1] \subseteq I \longrightarrow \mathbb{R}$ be a non-negative function. A function $f : K = [m, m+\eta(n, m)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is known as relative harmonic preinvex function with respect to an arbitrary function h and $\eta(., .)$, if

$$f\left(\frac{m(m+\eta(n,m))}{m+(1-t)\eta(n,m)}\right) \leq h(1-t)f(m)+h(t)f(n), \quad t\in[0,1], \quad \forall\ m,n\in K.$$

Remark 7

- *If* $t=\frac{1}{2}$ *, then we get*

$$f\left(\frac{2m(m+\eta(n,m))}{2m+\eta(n,m)}\right) \leq h\left(\frac{1}{2}\right)[f(m)+f(n)], \quad \forall\ m,n\in K.$$

  which is known as Jensen type relative harmonic preinvex function.

- *If* $h(t)=t$*, then relative harmonic preinvex function reduces to harmonic preinvex function.*

- *If* $h(t)=t^s$*, then the harmonic preinvex functions becomes Breckner type of* s*-harmonic preinvex functions.*

- *If* $h(t)=t^{-s}$*, then the harmonic preinvex functions becomes Godunova-Levin type of* s*-harmonic preinvex functions.*

3.1.2 s-Harmonic Preinvex Function

A function $f : K=[m,m+\eta(n,m)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ is known as s-harmonic preinvex function with respect to bifunction $\eta(.,.)$, if

$$f\left(\frac{m(m+\eta(n,m))}{m+(1-t)\eta(n,m)}\right) \leq (1-t)^s f(m)+t^s f(n), \quad t\in[0,1], \quad s\in(0,1], \quad \forall\ m,n\in K.$$

3.1.3 Godunova-Levin Type of s-Harmonic Preinvex Function

A function $f : K=[m,m+\eta(n,m)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ is known as Godunova-Levin type of s-harmonic preinvex function with respect to bifunction $\eta(.,.)$, if

$$f\left(\frac{m(m+\eta(n,m))}{m+(1-t)\eta(n,m)}\right) \leq (1-t)^{-s} f(m)+t^{-s} f(n), \quad t\in[0,1], \quad s\in(0,1], \quad \forall\ m,n\in K.$$

Remark 8

- *If* $s=0$*, then Godunova-Levin type of* s*-harmonic preinvex functions becomes harmonic* P*-preinvex functions.*

- *If* $s=1$*, then Godunova-Levin type of* s*-harmonic preinvex functions becomes Godunova-Levin type of harmonic preinvex functions.*

3.1.4 Harmonic P-preinvex Function

A function $f : K=[m,m+\eta(n,m)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ is known as harmonic P-preinvex function with respect to to bifunction $\eta(.,.)$, if

$$f\left(\frac{m(m+\eta(n,m))}{m+(1-t)\eta(n,m)}\right) \leq f(m)+f(n), \quad t\in[0,1], \quad \forall\ m,n\in K.$$

3.1.5 Condition C

Let $J \subset \mathbb{R}$ be an invex set with respect to the bifunction $\eta(.,.)$, then for any $m, n \in J$ and $t_1, t_2 \in [0,1]$, we have

$$\eta(n + t_2\eta(m,n), n + t_1\eta(m,n)) = (t_2 - t_1)\eta(m,n) \quad t \in [0,1], \quad \forall\ m, n \in J$$

3.1.6 Hermite-Hadamard Inequality for Harmonic Preinvex Function

A function $f : K = [m, m + \eta(n,m)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is harmonic preinvex function. If $f \in L[m, m + \eta(n,m)]$, then

$$f\left(\frac{2m(m + \eta(n,m))}{2m + \eta(n,m)}\right) \leq \frac{m(m + \eta(n,m))}{\eta(n,m)} \int_m^{m+\eta(n,m)} \frac{f(x)}{x^2} dx \leq \frac{f(m) + f(n)}{2}.$$

3.1.7 Hermite-Hadamard Inequality for Relative Harmonic Preinvex Function

A function $f : K = [m, m + \eta(n,m)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is relative harmonic preinvex function where $m, m + \eta(n,m) \in K$ with $m < m + \eta(n,m)$. If $f \in L[m, m + \eta(n,m)]$ and condition C holds,

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{2m(m + \eta(n,m))}{2m + \eta(n,m)}\right) \leq \frac{m(m + \eta(n,m))}{\eta(n,m)} \int_m^{m+\eta(n,m)} \frac{f(x)}{x^2} dx \leq [f(m) + f(n)] \int_0^1 h(t)dt.$$

3.1.8 Hermite-Hadamard Inequality for s-Harmonic Preinvex Function

A function $f : K = [m, m + \eta(n,m)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is s-harmonic preinvex function. If $f \in L[m, m + \eta(n,m)]$, then

$$2^{s-1} f\left(\frac{2m(m + \eta(n,m))}{2m + \eta(n,m)}\right) \leq \frac{m(m + \eta(n,m))}{\eta(n,m)} \int_m^{m+\eta(n,m)} \frac{f(x)}{x^2} dx \leq \frac{f(m) + f(n)}{s + 1}.$$

3.1.9 Hermite-Hadamard Inequality for s-Harmonic Godunova-Levin Preinvex Function

A function $f : K = [m, m + \eta(n,m)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is s-harmonic Godunova-Levin preinvex function. If $f \in L[m, m + \eta(n,m)]$, then

$$\frac{1}{2^{s+1}} f\left(\frac{2m(m + \eta(n,m))}{2m + \eta(n,m)}\right) \leq \frac{m(m + \eta(n,m))}{\eta(n,m)} \int_n^{m+\eta(n,m)} \frac{f(x)}{x^2} dx \leq \frac{f(m) + f(n)}{1 - s}.$$

3.1.10 Hermite-Hadamard Inequality for Harmonic P-preinvex Function

A function $f : K = [m, m + \eta(n,m)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is harmonic P-preinvex function. If $f \in L[m, m + \eta(n,m)]$, then

$$\frac{1}{2} f\left(\frac{2m(m + \eta(n,m))}{2m + \eta(n,m)}\right) \leq \frac{m(m + \eta(n,m))}{\eta(n,m)} \int_m^{m+\eta(n,m)} \frac{f(x)}{x^2} dx \leq f(m) + f(n).$$

3.1.11 Regularized Hypergeometric Function

Given a generalized hypergeometric or hypergeometric function ${}_pF_q(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;x)$, the corresponding regularized hypergeometric function is given by

$${}_p\tilde{F}_q(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;x) = \frac{{}_pF_q(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;x)}{(\Gamma(\beta_1)...\Gamma(\beta_q))}, \quad \textit{here } \Gamma(x) \textit{ is a gamma function.}$$

3.1.12 Appell Hypergeometric Function

In the product of two hypergeometric functions $F(\alpha;\beta;\gamma;x)$, $F(\alpha';\beta';\gamma';y)$, we obtain a double series , resulting in four kinds of functions which are shown as follows:

$$F_1(\alpha;\beta,\beta';\gamma;x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha)_{r+s}(\beta)_r(\beta')_s}{r!s!(\gamma)_{r+s}}x^r y^s, \qquad |x|,|y|<1$$

$$F_2(\alpha;\beta,\beta';\gamma,\gamma';x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha)_{r+s}(\beta)_r(\beta')_s}{r!s!(\gamma)_r(\gamma')_v}x^r y^s\ , \qquad |x|+|y|<1$$

$$F_3(\alpha,\alpha';\beta,\beta';\gamma;x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s}{r!s!(\gamma)_{r+s}}x^r y^s, \qquad |x|,|y|<1$$

$$F_4(\alpha;\beta;\gamma,\gamma';x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha)_{r+s}(\beta)_{r+s}}{r!s!(\gamma)_r(\gamma')_s}x^r y^s, \qquad |x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}<1.$$

3.1.13 Riemann-Liouville Integrals

Let $f\in L[u,v]$. The Riemann-Liouville Integrals $J^{\beta}_{u+}f$ and $J^{\beta}_{v-}f$ of order $\beta>0$ with $u\geq 0$ are given by

$$J^{\beta}_{u+}f(x) = \frac{1}{\Gamma(\beta)}\int_u^x (x-t)^{\beta-1}f(t)dt, \quad x>u$$

$$J^{\beta}_{v-}f(x) = \frac{1}{\Gamma(\beta)}\int_x^v (t-x)^{\beta-1}f(t)dt, \quad v>x$$

Here, $\Gamma(\beta)=\int_0^{+\infty}e^{-a}a^{\beta-1}da.$

If $\beta=0$, then $J^0_{u+}f(x)=J^0_{v-}f(x)=f(x)$

If $\beta=1$, then the fractional integral becomes the classical integral.

3.2 Main Results

Lemma 2 *Assuming that $f: M=[u,u+\eta(v,u)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ be differentiable on the interior, M^o of M where $f\in L_1[u,u+\eta(v,u)]$ for $u,u+\eta(v,u)\in M$ with $u<u+\eta(v,u)$, $\lambda,\alpha\in[0,1]$, $g(x)=\frac{1}{x}$ and $\beta\in(0,1]$ such that $(-1)^{\beta}\in\mathbb{R}$, then*

$$\begin{aligned}
&\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))\\
&:= -\Bigg[f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)[(1-\alpha)^{\beta}-(-1)^{\beta}\alpha^{\beta}-\lambda]+\lambda(1-\alpha)f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_1}\right)\alpha\lambda\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_o}\right)\\
&\quad -\frac{\Gamma(\beta+1)u^{\beta}\{u+\eta(v,u)\}^{\beta}}{\eta(v,u)^{\beta}}\left\{J^{\beta}_{\frac{u\alpha+(1-\alpha)\{u+\eta(v,u)\}}{u\{u+\eta(v,u)\}}-}f\circ g\left(\frac{1}{u+\eta(v,u)}\right)\right.\\
&\quad \left.+(-1)^{\beta}J^{\beta}_{\frac{u\alpha+(1-\alpha)\{u+\eta(v,u)\}}{u\{u+\eta(v,u)\}}+}f\circ g\left(\frac{1}{u}\right)\right\}\Bigg]\\
&= \quad u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt+\int_{1-\alpha}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}\right.
\end{aligned}$$

$$f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\Bigg]$$

for $t\in[0,1]$ *and* $\bar{A}_t=(1-t)u+t(u+\eta(v,u))$.

Proof. Let

$$\begin{aligned}
I_1 &= \int_0^{1-\alpha}\frac{t^\beta-\alpha\lambda}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\\
&= -\int_0^{1-\alpha}\frac{t^\beta-\alpha\lambda}{[(1-t)u+t(u+\eta(v,u))]^2}\frac{-u\eta(v,u)\{u+\eta(v,u)\}}{[(1-t)u+t(u+\eta(v,u))]^2}\\
&\quad\frac{[(1-t)u+t(u+\eta(v,u))]^2}{u\eta(v,u)\{u+\eta(v,u)\}}f'\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)dt\\
&= -\int_0^{1-\alpha}\frac{t^\beta-\alpha\lambda}{u\{u+\eta(v,u)\}\eta(v,u)}df\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)\\
&= -\frac{1}{u\{u+\eta(v,u)\}\eta(v,u)}\int_0^{1-\alpha}(t^\beta-\alpha\lambda)df\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)\\
&= -\frac{1}{u\{u+\eta(v,u)\}\eta(v,u)}\left[(t^\beta-\alpha\lambda)f\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)\Bigg|_0^{1-\alpha}\right.\\
&\quad\left.-\int_0^{1-\alpha}f\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)\beta t^{\beta-1}dt\right]\\
&= -\frac{1}{u\{u+\eta(v,u)\}\eta(v,u)}\left[\{(1-\alpha)^\beta-\alpha\lambda\}f\left(\frac{u\{u+\eta(v,u)\}}{\alpha u+(1-\alpha)(u+\eta(v,u))}\right)\right.\\
&\quad\left.+\alpha\lambda f\left(\frac{u\{u+\eta(v,u)\}}{u}\right)-\beta\int_0^{1-\alpha}t^{\beta-1}f\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)dt\right]. \qquad (3.1)
\end{aligned}$$

Setting $x=\frac{(1-t)u+t(u+\eta(v,u))}{u\{u+\eta(v,u)\}}$, so that $dx=\frac{\eta(v,u)}{u\{u+\eta(v,u)\}}dt$
For $0\le t\le 1-\alpha$, we have $\frac{1}{u+\eta(v,u)}\le x\le\frac{u\alpha+(1-\alpha)(u+\eta(v,u))}{u\{u+\eta(v,u)\}}$ and hence (3.1) becomes

$$\begin{aligned}
I_1 &= -\frac{1}{u\eta(v,u)\{u+\eta(v,u)\}}\left[\{(1-\alpha)^\beta-\alpha\lambda\}f\left(\frac{u\{u+\eta(v,u)\}}{\alpha u+(1-\alpha)(u+\eta(v,u))}\right)\right.\\
&\quad+\alpha\lambda f\left(\frac{u\{u+\eta(v,u)\}}{u}\right)-\beta\int_{\frac{1}{u+\eta(v,u)}}^{\frac{u\alpha+(1-\alpha)(u+\eta(v,u))}{u\{u+\eta(v,u)\}}}\left(\frac{xu\{u+\eta(v,u)\}-u}{\eta(v,u)}\right)^{\beta-1}\\
&\quad\left.f\left(\frac{1}{x}\right)\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}dx\right]\\
I_1 &= -\frac{1}{u\eta(v,u)\{u+\eta(v,u)\}}\left[\{(1-\alpha)^\beta-\alpha\lambda\}f\left(\frac{u\{u+\eta(v,u)\}}{\alpha u+(1-\alpha)(u+\eta(v,u))}\right)\right.\\
&\quad+\alpha\lambda f\left(\frac{u\{u+\eta(v,u)\}}{u}\right)-\frac{\beta u^\beta\{u+\eta(v,u)\}^\beta}{\{\eta(v,u)\}^\beta}\\
&\quad\left.\int_{\frac{1}{u+\eta(v,u)}}^{\frac{u\alpha+(1-\alpha)(u+\eta(v,u))}{u\{u+\eta(v,u)\}}}(f\circ g)(x)\left[x-\frac{1}{u+\eta(v,u)}\right]^{\beta-1}dx\right] \qquad \left(for\ \ \frac{1}{x}=g(x)\right)\\
I_1 &= -\frac{1}{u\eta(v,u)\{u+\eta(v,u)\}}\left[\{(1-\alpha)^\beta-\alpha\lambda\}f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right.\\
&\quad+\alpha\lambda f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_o}\right)-\frac{\beta u^\beta\{u+\eta(v,u)\}^\beta}{\{\eta(v,u)\}^\beta}\\
&\quad\left.\int_{\frac{1}{u+\eta(v,u)}}^{\frac{u\alpha+(1-\alpha)(u+\eta(v,u))}{u\{u+\eta(v,u)\}}}(f\circ g)(x)\left[x-\frac{1}{u+\eta(v,u)}\right]^{\beta-1}dx\right]
\end{aligned}$$

$$I_1 = -\frac{1}{u\eta(v,u)\{u+\eta(v,u)\}}\left[\{(1-\alpha)^\beta-\alpha\lambda\}f\ (\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)+\alpha\lambda f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_o}\right)$$

$$-\frac{\beta u^{\beta}\{u+\eta(v,u)\}^{\beta}}{\{\eta(v,u)\}^{\beta}}\left\{\Gamma(\beta)J^{\beta}_{\frac{u\alpha+(1-\alpha)(u+\eta(v,u))}{u\{u+\eta(v,u)\}}-}(f\circ g)\left(\frac{1}{u+\eta(v,u)}\right)\right\}\Bigg] \tag{3.2}$$

Let

$$\begin{aligned}
I_2 &= \int_{1-\alpha}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\\
&= -\int_{1-\alpha}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{[(1-t)u+t(u+\eta(v,u))]^2}\frac{-u\{u+\eta(v,u)\}\eta(v,u)}{[(1-t)u+t(u+\eta(v,u))]^2}\\
&\quad \frac{[(1-t)u+t(u+\eta(v,u))]^2}{u\{u+\eta(v,u)\}\eta(v,u)}f'\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)dt\\
&= -\int_{1-\alpha}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{u\eta(v,u)\{u+\eta(v,u)\}}df\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)\\
&= -\frac{1}{u\eta(v,u)\{u+\eta(v,u)\}}\int_{1-\alpha}^{1}((t-1)^{\beta}-\lambda(\alpha-1))df\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)\\
&= -\frac{1}{u\eta(v,u)\{u+\eta(v,u)\}}\Bigg[((t-1)^{\beta}-\lambda(\alpha-1))f\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)\Bigg|_{1-\alpha}^{1}\\
&\quad -\int_{1-\alpha}^{1}f\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)\beta(t-1)^{\beta-1}dt\Bigg]\\
&= -\frac{1}{u\eta(v,u)\{u+\eta(v,u)\}}\Bigg[-\{(-\alpha)^{\beta}+\lambda(1-\alpha)\}f\left(\frac{u\{u+\eta(v,u)\}}{(\alpha)u+(1-\alpha)(u+\eta(v,u))}\right)\\
&\quad +\lambda(1-\alpha)f\left(\frac{u\{u+\eta(v,u)\}}{u+\eta(v,u)}\right)-\beta\int_{1-\alpha}^{1}(t-1)^{\beta-1}f\left(\frac{u\{u+\eta(v,u)\}}{(1-t)u+t(u+\eta(v,u))}\right)dt\Bigg]
\end{aligned} \tag{3.3}$$

Setting $x=\frac{(1-t)u+t(u+\eta(v,u))}{u\{u+\eta(v,u)\}}$, so that $dx=\frac{\eta(v,u)}{u\{u+\eta(v,u)\}}dt$.
For $1-\alpha\le t\le 1$, we have $\frac{u\alpha+(1-\alpha)(u+\eta(v,u))}{u\{u+\eta(v,u)\}}\le x\le\frac{1}{u}$ and hence (3.3) becomes

$$\begin{aligned}
I_2 &= -\frac{1}{u\eta(v,u)\{u+\eta(v,u)\}}\Big[-\{(-\alpha)^{\beta}-\lambda(1-\alpha)\}\\
&\quad f\left(\frac{u\{u+\eta(v,u)\}}{\alpha u+(1-\alpha)(u+\eta(v,u))}\right)+\lambda(1-\alpha)f\left(\frac{u\{u+\eta(v,u)\}}{u+\eta(v,u)}\right)\\
&\quad -\beta\int_{\frac{u\alpha+(1-\alpha)(u+\eta(v,u))}{u\{u+\eta(v,u)\}}}^{\frac{1}{u}}\left(\frac{xu\{u+\eta(v,u)\}-u}{\eta(v,u)}-1\right)^{\beta-1}f\left(\frac{1}{x}\right)\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}du\Bigg]\\
I_2 &= -\frac{1}{u\eta(v,u)\{u+\eta(v,u)\}}\Big[-\{(-\alpha)^{\beta}-\lambda(1-\alpha)\}\\
&\quad f\left(\frac{u\{u+\eta(v,u)\}}{\alpha u+(1-\alpha)(u+\eta(v,u))}\right)+\lambda(1-\alpha)f\left(\frac{u\{u+\eta(v,u)\}}{u+\eta(v,u)}\right)\\
&\quad -\frac{\beta u^{\beta}}{\{u+\eta(v,u)\}^{\beta}}\{\eta(v,u)\}^{\beta}\int_{\frac{u\alpha+(1-\alpha)(u+\eta(v,u))}{u\{u+\eta(v,u)\}}}^{\frac{1}{u}}(f\circ g)(x)\left[x-\frac{1}{u}\right]^{\beta-1}dx\Bigg]\left(for\ \ \frac{1}{x}=g(x)\right)\\
I_2 &= -\frac{1}{u\eta(v,u)\{u+\eta(v,u)\}}\Big[-\{(-\alpha)^{\beta}-\lambda(1-\alpha))\\
&\quad f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)+\lambda(1-\alpha)f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_1}\right)\\
&\quad -\frac{\beta u^{\beta}\{u+\eta(v,u)\}^{\beta}}{\{\eta(v,u)\}^{\beta}}\int_{\frac{u\alpha+(1-\alpha)(u+\eta(v,u))}{u\{u+\eta(v,u)\}}}^{\frac{1}{u}}(f\circ g)(x)\left[x-\frac{1}{u}\right]^{\beta-1}dx\Bigg]\\
I_2 &= -\frac{1}{u\eta(v,u)\{u+\eta(v,u)\}}\Big[-\{(1-\alpha)^{\beta}-\lambda(1-\alpha)\}\\
&\quad f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)+\lambda(1-\alpha)f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_1}\right)-(-1)^{\beta-1}\\
&\quad \frac{\beta u^{\beta}\{u+\eta(v,u)\}^{\beta}}{\{\eta(v,u)\}^{\beta}}\left\{\Gamma(\beta)J^{\beta}_{\frac{u\alpha+(1-\alpha)(u+\eta(v,u))}{u\{u+\eta(v,u)\}}+}(f\circ g)\left(\frac{1}{u}\right)\right\}\Bigg]
\end{aligned} \tag{3.4}$$

Adding Equations (3.2) and (3.4), we have

$$\begin{aligned}\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u)) &= -\Bigg[f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)[(1-\alpha)^{\beta}-\alpha\lambda-(-1)^{\beta}\alpha^{\beta}-\lambda+\alpha\lambda] \\ &\quad +\lambda(1-\alpha)f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_1}\right)+\alpha\lambda\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_o}\right) \\ &\quad -\frac{\Gamma(\beta+1)u^{\beta}\{u+\eta(v,u)\}^{\beta}}{\eta(v,u)^{\beta}}\Bigg\{J^{\beta}_{\frac{u\alpha+(1-\alpha)\{u+\eta(v,u)\}}{u\{u+\eta(v,u)\}}-}f\circ g\left(\frac{1}{u+\eta(v,u)}\right) \\ &\quad +(-1)^{\beta}J^{\beta}_{\frac{u\alpha+(1-\alpha)\{u+\eta(v,u)\}}{u\{u+\eta(v,u)\}}+}f\circ g\left(\frac{1}{u}\right)\Bigg\}\Bigg] \\ &= u\eta(v,u)\{u+\eta(v,u)\}\Bigg[\int_0^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt \\ &\quad +\int_{1-\alpha}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\Bigg]\end{aligned}$$

$$\begin{aligned}\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u)) &:= -\Bigg[f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)[(1-\alpha)^{\beta}-(-1)^{\beta}\alpha^{\beta}-\lambda] \\ &\quad +\lambda(1-\alpha)f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_1}\right)+\alpha\lambda\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_o}\right) \\ &\quad -\frac{\Gamma(\beta+1)u^{\beta}\{u+\eta(v,u)\}^{\beta}}{\eta(v,u)^{\beta}}\Bigg\{J^{\beta}_{\frac{u\alpha+(1-\alpha)\{u+\eta(v,u)\}}{u\{u+\eta(v,u)\}}-}f\circ g\left(\frac{1}{u+\eta(v,u)}\right) \\ &\quad +(-1)^{\beta}J^{\beta}_{\frac{u\alpha+(1-\alpha)\{u+\eta(v,u)\}}{u\{u+\eta(v,u)\}}+}f\circ g\left(\frac{1}{u}\right)\Bigg\}\Bigg] \\ &= u\eta(v,u)\{u+\eta(v,u)\}\Bigg[\int_0^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt \\ &\quad +\int_{1-\alpha}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\Bigg]\end{aligned}$$

□

Remark 9 (a) *If $\lambda=0$, $\alpha=\frac{1}{2}$ and $\beta=1$, then Lemma 2 reduces to the following result*

$$\begin{aligned}&-\left[f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{\frac{1}{2}}}\right)-\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{\frac{1}{u+\eta(v,u)}}^{\frac{1}{u}}f\left(\frac{1}{x}\right)dx\right] \\ &= u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\right. \\ &\qquad \left.+\int_{\frac{1}{2}}^{1}\frac{(t-1)}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\right] \qquad (3.5)\end{aligned}$$

Setting $z=\frac{1}{x}$, so that $dx=\frac{-1}{z^2}dz$

For $\frac{1}{u+\eta(v,u)}\leq x\leq\frac{1}{u}$, we have $u+\eta(v,u)\leq z\leq u$ and hence (3.5) becomes

$$\begin{aligned}&-\left[f\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{\frac{1}{2}}}\right)-\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz\right] \\ &= u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\right. \\ &\qquad \left.+\int_{\frac{1}{2}}^{1}\frac{(t-1)}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\right] \qquad (3.6)\end{aligned}$$

Putting the value of $\bar{A}_{\frac{1}{2}}$ *in* (3.6), *we have*

$$-\left[f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)-\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz\right]$$
$$=u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\frac{(t-1)}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\right]$$

$$\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz-f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)$$
$$=u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\frac{(t-1)}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\right]. \tag{3.7}$$

(b) *If* $\lambda=1$, $\alpha=\frac{1}{2}$ *and* $\beta=1$, *then Lemma 2 reduces to the following result*

$$-\left[\frac{f(u)+f(u+\eta(v,u))}{2}-\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{\frac{1}{u+\eta(v,u)}}^{\frac{1}{u}}f\left(\frac{1}{x}\right)dx\right]$$
$$=u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{(t-\frac{1}{2})}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\frac{(t-\frac{1}{2})}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\right]$$

$$-\left[\frac{f(u)+f(u+\eta(v,u))}{2}-\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz\right]$$
$$=u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1}\frac{(t-\frac{1}{2})}{(\bar{A}_t)^2}f'\left(\frac{u\{v+\eta(v,u)\}}{\bar{A}_t}\right)dt\right]$$

$$\frac{f(u)+f(u+\eta(v,u))}{2}-\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz$$
$$=u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1}\frac{(\frac{1}{2}-t)}{(\bar{A}_t)^2}f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)dt\right].$$

Now we establish new integral inequalities of Hermite-Hadamard type for relative harmonically preinvex functions.

Theorem 4 *Assuming that* $f:M=[u,u+\eta(v,u)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f'\in L[u+\eta(v,u)]$ *for* $u,u+\eta(v,u)\in M$ *with* $u<u+\eta(v,u)$, $\beta\in(0,1]$ *and* $\lambda,\alpha\in[0,1]$. *If* $|f'|^\mu$ *is relative harmonically preinvex on* M *for* $\mu>1$ *with* $\frac{1}{\gamma}+\frac{1}{\mu}=1$, *we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}}\leq 1-\alpha\leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))|\leq u\eta(v,u)\{u+\eta(v,u)\}[(k_1(\lambda,\alpha,\beta,u,v)$$

$$+k_2(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(k_7(\lambda,\alpha,\beta,u,v,h)$$
$$+k_8(\lambda,\alpha,\beta,u,v,h)|f'(u)|^{\mu}+(k_9(\lambda,\alpha,\beta,u,v,h)$$
$$+k_{10}(\lambda,\alpha,\beta,u,v,h))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}+(k_5(\lambda,\alpha,\beta,u,v)$$
$$+k_6(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(k_{15}(\lambda,\alpha,\beta,u,v,h)$$
$$+k_{16}(\lambda,\alpha,\beta,u,v,h))|f'(u)|^{\mu}+(k_{17}(\lambda,\alpha,\beta,u,v,h)$$
$$+k_{18}(\lambda,\alpha,\beta,u,v,h))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\\ & [(k_1(\lambda,\alpha,\beta,u,v)+k_2(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\\ & \{(k_7(\lambda,\alpha,\beta,u,v,h)+k_8(\lambda,\alpha,\beta,u,v,h))|f'(u)|^{\mu}\\ & +(k_9(\lambda,\alpha,\beta,u,v,h)+k_{10}(\lambda,\alpha,\beta,u,v,h))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}\\ & +(k_4(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(k_{13}(\lambda,\alpha,\beta,u,v,h)|f'(u)|^{\mu}\\ & +k_{14}(\lambda,\alpha,\beta,u,v,h)|f'(v)|^{\mu})\}^{\frac{1}{\mu}}].\end{aligned}$$

(c) *If* $1-\alpha \leq (\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\\ & [(k_3(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(k_{11}(\lambda,\alpha,\beta,u,v,h)|f'(u)|^{\mu}\\ & +k_{12}(\lambda,\alpha,\beta,u,v,h)|f'(v)|^{\mu})\}^{\frac{1}{\mu}}+(k_5(\lambda,\alpha,\beta,u,v)\\ & +k_6(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(k_{15}(\lambda,\alpha,\beta,u,v,h)\\ & +k_{16}(\lambda,\alpha,\beta,u,v,h))|f'(u)|^{\mu}+(k_{17}(\lambda,\alpha,\beta,u,v,h)\\ & +k_{18}(\lambda,\alpha,\beta,u,v,h))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].\end{aligned}$$

Proof. By using Lemma 2 and power mean integral inequality, we have

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right.\\ & \left.+\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right]\\ \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\left(\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}\right.\\ & \left.\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}}+\left(\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\\ & \left(\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}} \qquad (3.8)\end{aligned}$$

(a) (i) If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}dt$$

$$\begin{aligned} &= \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} dt + \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta - \alpha\lambda}{(\bar{A}_t)^2} dt \\ &= k_1(\lambda, \alpha, \beta, u, v) + k_2(\lambda, \alpha, \beta, u, v) \end{aligned} \quad (3.9)$$

where

$$k_1(\lambda, \alpha, \beta, u, v) := \frac{(\alpha\lambda^{\frac{1}{\beta}})^{\beta+1}\beta\ {}_2F_1[1, 1+\beta, 2+\beta, -\frac{c\alpha\lambda^{\frac{1}{\beta}}}{u}]}{u^2(1+\beta)} + \frac{\alpha\lambda^{\frac{1}{\beta}}(-((\alpha\lambda^{\frac{1}{\beta}})^\beta - \alpha\lambda))}{u(u + c\alpha\lambda^{\frac{1}{\beta}})}$$

$$\begin{aligned} k_2(\lambda, \alpha, \beta, u, v) := &\frac{c(1-\alpha)^{1+\beta} + u\alpha\lambda}{uc(u + c - c\alpha)} - \frac{c(\alpha\lambda^{\frac{1}{\beta}})^{1+\beta} + u\alpha\lambda}{uc(u + c\alpha\lambda^{\frac{1}{\beta}})} \\ &+ \frac{(-u + c(\alpha - 1))(1-\alpha)^{1+\beta}\beta {}_2F_1[1, 1+\beta, 2+\beta, \frac{c(\alpha-1)}{u}]}{u^2(u + c - c\alpha)(1+\beta)} \\ &+ \frac{(\alpha\lambda^{\frac{1}{\beta}})^{1+\beta}\beta {}_2F_1[1, 1+\beta, 2+\beta, -\frac{c\alpha\lambda^{\frac{1}{\beta}}}{u}]}{u^2(1+\beta)} \end{aligned}$$

(*ii*) If $(\alpha\lambda)^{\frac{1}{\beta}} \geq 1 - \alpha$, then

$$\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} dt = \int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} dt = k_3(\lambda, \alpha, \beta, u, v). \quad (3.10)$$

where

$$\begin{aligned} k_3(\lambda, \alpha, \beta, u, v) := &\frac{(-1+\alpha)(u(1+\beta)((1-\alpha)^\beta - \alpha\lambda))}{u^2(u + c - c\alpha)(1+\beta)} \\ &- \frac{(-1+\alpha)(1-\alpha)^\beta(u + c - c\alpha)\beta {}_2F_1[1, 1+\beta, 2+\beta, \frac{c(-1+\alpha)}{u}])}{u^2(u + c - c\alpha)(1+\beta)} \end{aligned}$$

(b) (*i*) If $1 + (\lambda(\alpha - 1))^{\frac{1}{\beta}} \leq 1 - \alpha$, then

$$\begin{aligned} &\int_{1-\alpha}^1 \frac{|(t-1)^\beta - \lambda(\alpha - 1)|}{(\bar{A}_t)^2} dt \\ &= \int_{1-\alpha}^1 \frac{(t-1)^\beta - \lambda(\alpha - 1)}{(\bar{A}_t)^2} dt \\ &= k_4(\lambda, \alpha, \beta, u, v). \end{aligned} \quad (3.11)$$

where

$$k_4(\lambda, \alpha, \beta, u, v) := \int_{1-\alpha}^1 \frac{(t-1)^\beta - \lambda(\alpha - 1)}{(1-t)u + t(u + c)} dt$$

(*ii*) If $1 + (\lambda(\alpha - 1))^{\frac{1}{\beta}} \geq 1 - \alpha$, then

$$\begin{aligned} &\int_{1-\alpha}^1 \frac{|(t-1)^\beta - \lambda(\alpha - 1)|}{(\bar{A}_t)^2} dt \\ &= \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^\beta - \lambda(\alpha - 1))}{(\bar{A}_t)^2} dt \\ &\quad + \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^1 \frac{(t-1)^\beta - \lambda(\alpha - 1)}{(\bar{A}_t)^2} dt \end{aligned}$$

$$= \quad k_5(\lambda,\alpha,\beta,u,v)+k_6(\lambda,\alpha,\beta,u,v). \tag{3.12}$$

where

$$k_5(\lambda,\alpha,\beta,u,v):=\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(1-t)u+t(u+c))}dt$$

$$k_6(\lambda,\alpha,\beta,u,v):=\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(1-t)u+t(u+c)}dt$$

Since $|f'|^{\mu}$ be relative harmonically preinvex on the interval $[u,u+\eta(v,u)]$ with respect to an arbitrary nonnegative function h and for $\mu>1$, as $t\in[0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^{\mu}\le h(t)|f'(u)|^{\mu}+h(1-t)|f'(v)|^{\mu}.$$

hence, by calculation, we get

(c) (i) If $(\alpha\lambda)^{\frac{1}{\beta}}\le 1-\alpha$, then

$$\begin{aligned}
&\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\\
\le\ &\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(\bar{A}_t)^2}[h(t)|f'(u)|^{\mu}+h(1-t)|f'(v)|^{\mu}]dt\\
&+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(\bar{A}_t)^2}[h(t)|f'(u)|^{\mu}+h(1-t)|f'(v)|^{\mu}]dt\\
=\ &\left[\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(\bar{A}_t)^2}h(t)dt+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(\bar{A}_t)^2}h(t)dt\right]|f'(u)|^{\mu}\\
&+\left[\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(\bar{A}_t)^2}h(1-t)dt+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(\bar{A}_t)^2}h(1-t)dt\right]|f'(v)|^{\mu}\\
=\ &[k_7(\lambda,\alpha,\beta,u,v,h)+k_8(\lambda,\alpha,\beta,u,v,h)]|f'(u)|^{\mu}\\
&+[k_9(\lambda,\alpha,\beta,u,v,h)+k_{10}(\lambda,\alpha,\beta,u,v,h)]|f'(v)|^{\mu}.
\end{aligned}\tag{3.13}$$

where

$$k_7(\lambda,\alpha,\beta,u,v,h):=\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(u(1-t)+(u+c)t)^2}h(t)dt$$

$$k_8(\lambda,\alpha,\beta,u,v,h):=\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(u(1-t)+(u+c)t)^2}h(t)dt$$

$$k_9(\lambda,\alpha,\beta,u,v,h):=\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(u(1-t)+(u+c)t)^2}h(1-t)dt$$

$$k_{10}(\lambda,\alpha,\beta,u,v,h):=\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(u(1-t)+(u+c)t)^2}h(1-t)dt$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}}\ge 1-\alpha$, then

$$\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt$$

$$\begin{aligned}
&\leq \int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2}[h(t)|f'(u)|^\mu + h(1-t)|f'(v)|^\mu]dt \\
&= \int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2}h(t)dt|f'(u)|^\mu + \int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2}h(1-t)dt|f'(v)|^\mu \\
&= k_{11}(\lambda,\alpha,\beta,u,v,h)|f'(u)|^\mu + k_{12}(\lambda,\alpha,\beta,u,v,h)|f'(v)|^\mu. \qquad (3.14)
\end{aligned}$$

where

$$k_{11}(\lambda,\alpha,\beta,u,v,h) := \int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(u(1-t)+(u+c)t)^2}h(t)dt$$

$$k_{12}(\lambda,\alpha,\beta,u,v,h) := \int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(u(1-t)+(u+c)t)^2}h(1-t)dt$$

(d) (*i*) If $1 + (\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^{1} \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt \\
&\leq \int_{1-\alpha}^{1} \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2}[h(t)|f'(u)|^\mu + h(1-t)|f'(v)|^\mu]dt \\
&= \int_{1-\alpha}^{1} \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2}h(t)dt|f'(u)|^\mu \\
&\quad + \int_{1-\alpha}^{1} \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2}h(1-t)dt|f'(v)|^\mu \\
&= k_{13}(\lambda,\alpha,\beta,u,v,h)|f'(u)|^\mu + k_{14}(\lambda,\alpha,\beta,u,v,h)|f'(v)|^\mu. \qquad (3.15)
\end{aligned}$$

where

$$k_{13}(\lambda,\alpha,\beta,u,v,h) := \int_{1-\alpha}^{1} \frac{(t-1)^\beta - \lambda(\alpha-1)}{(u(1-t)+(u+c)t)^2}h(t)dt$$

$$k_{14}(\lambda,\alpha,\beta,u,v,h) := \int_{1-\alpha}^{1} \frac{(t-1)^\beta - \lambda(\alpha-1)}{(u(1-t)+(u+c)t)^2}h(1-t)dt$$

(*ii*) If $1 + (\lambda(\alpha-1))^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^{1} \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt \\
&\leq \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^\beta - \lambda(\alpha-1))}{(\bar{A}_t)^2}[h(t)|f'(u)|^\mu + h(1-t)|f'(v)|^\mu]dt \\
&\quad + \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2}[h(t)|f'(u)|^\mu + h(1-t)|f'(v)|^\mu]dt \\
&= \left[\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^\beta - \lambda(\alpha-1))}{(\bar{A}_t)^2}h(t)dt \right.\\
&\quad \left. + \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2}h(t)dt\right]|f'(u)|^\mu \\
&\quad + \left[\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^\beta - \lambda(\alpha-1))}{(\bar{A}_t)^2}h(1-t)dt \right.\\
&\quad \left. + \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2}h(1-t)dt\right]|f'(v)|^\mu
\end{aligned}$$

$$= [k_{15}(\lambda,\alpha,\beta,u,v,h)+k_{16}(\lambda,\alpha,\beta,u,v,h)]|f'(u)|^{\mu}$$
$$+[k_{17}(\lambda,\alpha,\beta,u,v,h)+k_{18}(\lambda,\alpha,\beta,u,v,h)]|f'(v)|^{\mu}. \quad (3.16)$$

where

$$k_{15}(\lambda,\alpha,\beta,u,v,h) := \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(u(1-t)+(u+c)t)^2}h(t)dt$$

$$k_{16}(\lambda,\alpha,\beta,u,v,h) := \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(u(1-t)+(u+c)t)^2}h(t)dt$$

$$k_{17}(\lambda,\alpha,\beta,u,v,h) := \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(u(1-t)+(u+c)t)^2}h(1-t)dt$$

$$k_{18}(\lambda,\alpha,\beta,u,v,h) := \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(u(1-t)+(u+c)t)^2}h(1-t)dt$$

Where $c=\eta(u,v)$. By substituting (3.9) to (3.16) in equation (3.8) gives the required result. □

Corollary 3 *Assuming that* $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ *is differentiable the interior,* M^o *of* M *where* $f' \in L_1[u, u+\eta(v,u)]$ *for* $u, u+\eta(v,u) \in M$ *with* $u < u+\eta(v,u)$, $\beta = 1$ *and* $\lambda,\alpha \in [0,1]$. *If* $|f'|^{\mu}$ *is relative harmonically preinvex on* M *for* $\mu > 1$ *with* $\frac{1}{\gamma}+\frac{1}{\mu}=1$, *we have*

(a) *If* $\alpha\lambda \le 1-\alpha \le 1+\lambda(\alpha-1)$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,1,u,u+\eta(v,u))| \le\ & u\eta(v,u)\{u+\eta(v,u)\}\\ & [(l_1(\lambda,\alpha,1,u,v)+l_2(\lambda,\alpha,1,u,v))^{\frac{1}{\gamma}}\\ & \{(l_7(\lambda,\alpha,1,u,v,h)+l_8(\lambda,\alpha,1,u,v,h))|f'(u)|^{\mu}\\ & +(l_9(\lambda,\alpha,1,u,v,h)+l_{10}(\lambda,\alpha,1,u,v,h))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}\\ & +(l_5(\lambda,\alpha,1,u,v)+l_6(\lambda,\alpha,1,u,v))^{\frac{1}{\gamma}}\\ & \{(l_{15}(\lambda,\alpha,1,u,v,h)+l_{16}(\lambda,\alpha,1,u,v,h))|f'(u)|^{\mu}\\ & +(l_{17}(\lambda,\alpha,1,u,v,h)+l_{18}(\lambda,\alpha,1,u,v,h))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].\end{aligned}$$

(b) *If* $\alpha\lambda \le 1+\lambda(\alpha-1) \le 1-\alpha$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,1,u,u+\eta(v,u))| \le\ & u\eta(v,u)\{u+\eta(v,u)\}[(l_1(\lambda,\alpha,1,u,v)\\ & +l_2(\lambda,\alpha,1,u,v))^{\frac{1}{\gamma}}\{(l_7(\lambda,\alpha,1,u,v,h)\\ & +l_8(\lambda,\alpha,1,u,v,h))|f'(u)|^{\mu}+(l_9(\lambda,\alpha,1,u,v,h)\\ & +l_{10}(\lambda,\alpha,1,u,v,h))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}\\ & +(l_4(\lambda,\alpha,1,u,v))^{\frac{1}{\gamma}}\{(l_{13}(\lambda,\alpha,1,u,v,h)|f'(u)|^{\mu}\\ & +l_{14}(\lambda,\alpha,1,u,v,h)|f'(v)|^{\mu})\}^{\frac{1}{\mu}}].\end{aligned}$$

(c) *If* $1-\alpha \le \alpha\lambda \le 1+\lambda(\alpha-1)$, *then*

$$|\Psi_f(\lambda,\alpha,1,u,u+\eta(v,u))| \le u\eta(v,u)\{u+\eta(v,u)\}[(l_3(\lambda,\alpha,1,u,v))^{\frac{1}{\gamma}}$$

$$\{(l_{11}(\lambda,\alpha,1,u,v,h)|f'(u)|^{\mu}+l_{12}(\lambda,\alpha,1,u,v,h)$$
$$|f'(v)|^{\mu})\}^{\frac{1}{\mu}}+(l_5(\lambda,\alpha,1,u,v)+l_6(\lambda,\alpha,1,u,v))^{\frac{1}{\gamma}}$$
$$\{(l_{15}(\lambda,\alpha,1,u,v,h)+l_{16}(\lambda,\alpha,1,u,v,h))|f'(u)|^{\mu}$$
$$+(l_{17}(\lambda,\alpha,1,u,v,h)+l_{18}(\lambda,\alpha,1,u,v,h))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].$$

where

$$l_1(\lambda,\alpha,1,u,v) := \frac{u+c\alpha\lambda+u\log(u)}{uc^2}-\frac{u+c\alpha\lambda+(u+c\alpha\lambda)\log(u+c\alpha\lambda)}{c^2(u+c\alpha\lambda)}$$

$$l_2(\lambda,\alpha,1,u,v) := -\frac{-u-c\alpha\lambda-(u+c-c\alpha)\log(u+c-c\alpha)}{c^2(u+c-c\alpha)} - \frac{u+c\alpha\lambda+(u+c\alpha\lambda)\log(u+c\alpha\lambda)}{c^2(u+c\alpha\lambda)}$$

$$l_3(\lambda,\alpha,1,u,v) := \frac{u+c\alpha\lambda+u\log(u)}{uc^2}-\frac{u+c\alpha\lambda+(u+c-c\alpha)\log(u+c-c\alpha)}{c^2(u+c-c\alpha)}$$

$$l_4(\lambda,\alpha,1,u,v) := \frac{u+c-c\lambda+c\alpha\lambda+(c+u)\log(c+u)}{c^2(c+u)} + \frac{-u+c(-1+\lambda-\alpha\lambda)-(u+c-c\alpha)\log(u+c-c\alpha)}{c^2(u+c-c\alpha)}$$

$$l_5(\lambda,\alpha,1,u,v) := \frac{u+c-c\lambda+c\alpha\lambda+(u+c-c\alpha)\log(u+c-c\alpha)}{c^2(u+c-c\alpha)} - \frac{1+\log(u+c-c\lambda+c\alpha\lambda)}{c^2}$$

$$l_6(\lambda,\alpha,1,u,v) := \frac{c(-1+\alpha)\lambda+(u+c)\log(\frac{u+c}{u+c+c(-1+\alpha)\lambda})}{c^2(u+c)}$$

$$l_7(\lambda,\alpha,1,u,v,h) := \int_0^{\alpha\lambda}\frac{-(t-\alpha\lambda)}{(u(1-t)+(u+c)t))^2}h(t)dt$$

$$l_8(\lambda,\alpha,1,u,v,h) := \int_{\alpha\lambda}^{1-\alpha}\frac{t-\alpha\lambda}{(u(1-t)+(u+c)t))^2}h(t)dt$$

$$l_9(\lambda,\alpha,1,u,v,h) := \int_0^{\alpha\lambda}\frac{-(t-\alpha\lambda)}{(u(1-t)+(u+c)t)^2}h(1-t)dt$$
$$l_{10}(\lambda,\alpha,1,u,v,h) := \int_{\alpha\lambda}^{1-\alpha}\frac{t-\alpha\lambda}{(u(1-t)+(u+c)t)^2}h(1-t)dt$$

$$l_{11}(\lambda,\alpha,1,u,v,h) := \int_0^{1-\alpha}\frac{-(t-\alpha\lambda)}{(u(1-t)+(u+c)t)^2}h(t)dt$$

$$l_{12}(\lambda,\alpha,1,u,v,h) := \int_0^{1-\alpha}\frac{-(t-\alpha\lambda)}{(u(1-t)+(u+c)t)^2}h(1-t)dt$$

$$l_{13}(\lambda,\alpha,1,u,v,h) := \int_{1-\alpha}^{1}\frac{(t-1)-\lambda(\alpha-1)}{(u(1-t)+(u+c)t)^2}h(t)dt$$

$$l_{14}(\lambda,\alpha,1,u,v,h) := \int_{1-\alpha}^{1} \frac{(t-1)-\lambda(\alpha-1)}{(u(1-t)+(u+c)t)^2} h(1-t)dt$$

$$l_{15}(\lambda,\alpha,1,u,v,h) := \int_{1-\alpha}^{1+(\lambda(\alpha-1))} \frac{-((t-1)-\lambda(\alpha-1))}{((u(1-t)+(u+c)t)^2} h(t)dt$$

$$l_{16}(\lambda,\alpha,1,u,v,h) := \int_{1+(\lambda(\alpha-1))}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(u(1-t)+(u+c)t)^2} h(t)dt$$

$$l_{17}(\lambda,\alpha,1,u,v,h) := \int_{1-\alpha}^{1+(\lambda(\alpha-1))} \frac{-((t-1)-\lambda(\alpha-1))}{(u(1-t)+(u+c)t)^2} h(1-t)dt$$

$$l_{18}(\lambda,\alpha,1,u,v,h) := \int_{1+(\lambda(\alpha-1))}^{1} \frac{(t-1)-\lambda(\alpha-1)}{(u(1-t)+(u+c)t)^2} h(1-t)dt$$

Remark 10 *If $\beta = 1$, $h(t) = t$ and $\eta(v,u) = v - u$, then the Theorem 4 reduces to the Theorem 1. This shows that the class of relative harmonic preinvex functions becomes the class of harmonic convex functions.*

Corollary 4 *Assuming that $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[u, u+\eta(v,u)]$ for $u, u+\eta(v,u) \in M$ with $u < u+\eta(v,u)$. If $|f'|^{\mu}$ is relative harmonically preinvex on M for $\mu > 1$ with $\frac{1}{\gamma} + \frac{1}{\mu} = 1$, then*

$$\left| \frac{u\{u+\eta(v,u)\}}{\eta(v,u)} \int_{u}^{u+\eta(v,u)} \frac{f(z)}{z^2} dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right) \right|$$

$$\leq u\eta(v,u)\ \{u+\eta(v,u)\}\ [\ s_5^{\frac{1}{\gamma}}\ (\lambda,u,v)\ \{s_1(\lambda,u,v,h)|f'(u)|^{\mu}$$

$$+ s_2(\lambda,u,v,h)|f'(v)|^{\mu}\}^{\frac{1}{\mu}} + s_6^{\frac{1}{\gamma}}(\lambda,u,v)\{s_3(\lambda,u,v,h)|f'(u)|^{\mu}$$

$$+ s_4(\lambda,u,v,h)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].$$

where

$$s_1(\lambda,u,v,h) := \int_0^{\frac{1}{2}} \frac{th(t)}{(u(1-t)+(u+c)t)^2} dt$$

$$s_2(\lambda,u,v,h) := \int_0^{\frac{1}{2}} \frac{th(1-t)}{(u(1-t)+(u+c)t)^2} dt$$

$$s_3(\lambda,u,v,h) := \int_{\frac{1}{2}}^{1} \frac{(1-t)h(t)}{(u(1-t)+(u+c)t)^2} dt$$

$$s_4(\lambda,u,v,h) := \int_{\frac{1}{2}}^{1} \frac{(1-t)h(1-t)}{(u(1-t)+(u+c)t)^2} dt$$

$$s_5(\lambda,u,v) := \frac{-\frac{c}{2u+c} - \log(u) + \log(u+\frac{c}{2})}{c^2}$$

$$s_6(\lambda,u,v) := \frac{\frac{c}{2u+c} + \log(u+\frac{c}{2} - \log(u+c))}{c^2}$$

Proof. From (3.7), we have

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\Big|dt + \int_{\frac{1}{2}}^1\frac{|t-1|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right].$$

By power mean integral inequality, we have

$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt\right)^{\frac{1}{\mu}}\right]$$

Since $|f'|^\mu$ be relative harmonically preinvex on the interval $[u, u+\eta(v,u)]$ with respect to an arbitrary nonnegative function h and for $\mu \in (1,\infty)$, as $t \in [0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^\mu \leq h(t)|f'(u)|^\mu + h(1-t)|f'(v)|^\mu$$

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}[h(t)|f'(u)|^\mu + h(1-t)|f'(v)|^\mu]dt\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(\bar{A}_t)^2}[h(t)|f'(u)|^\mu + h(1-t)|f'(v)|^\mu]dt\right)^{\frac{1}{\mu}}\right]$$
$$= u\eta(v,u)\{u+\eta(v,u)\}\left[\left(\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{th(t)}{(\bar{A}_t)^2}dt|f'(u)|^\mu + \int_0^{\frac{1}{2}}\frac{th(1-t)}{(\bar{A}_t)^2}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1\frac{-(t-1)}{(\bar{A}_t)^2}dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^1\frac{-(t-1)h(t)}{(\bar{A}_t)^2}dt|f'(u)|^\mu + \int_{\frac{1}{2}}^1\frac{-(t-1)h(1-t)}{(\bar{A}_t)^2}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}\right]$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}[s_5^{\frac{1}{\gamma}}(\lambda,u,v)\{s_1(\lambda,u,v,h)|f'(u)|^\mu + s_2(\lambda,u,v,h)|f'(v)|^\mu\}^{\frac{1}{\mu}}$$
$$+s_6^{\frac{1}{\gamma}}(\lambda,u,v)\{s_3(\lambda,u,v,h)|f'(u)|^\mu + s_4(\lambda,u,v,h)|f'(v)|^\mu\}^{\frac{1}{\mu}}];\ where\ \ c=\eta(v,u).$$

□

Theorem 5 *Assuming that $f: M=[u,u+\eta(v,u)] \subseteq \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[u,u+\eta(v,u)]$ for $u, u+\eta(v,u) \in M$ with $u < u+\eta(v,u), \beta \in (0,1]$ and $\lambda, \alpha \in [0,1]$. If $|f'|^\mu$ is relative harmonically preinvex on M for $\mu > 1$, we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}\Big[(k_{19}(\lambda,\alpha,\beta,u,v,\gamma)+k_{20}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}$$

$$\left((1-\alpha)\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(v)|^{\mu}\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}$$
$$+(k_{23}(\lambda,\alpha,\beta,u,v,\gamma)+k_{24}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}$$
$$\left.\left(\alpha\left\{|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}\right].$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}}\leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}\leq 1-\alpha$, *then*

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}\Big[(k_{19}(\lambda,\alpha,\beta,u,v,\gamma)+k_{20}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}$$
$$\left((1-\alpha)\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(v)|^{\mu}\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}$$
$$+(k_{22}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}$$
$$\left.\left(\alpha\left\{|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}\right].$$

(c) *If* $1-\alpha\leq(\alpha\lambda)^{\frac{1}{\beta}}\leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}\Big[(k_{21}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}$$
$$\left((1-\alpha)\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(v)|^{\mu}\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}$$
$$+(k_{23}(\lambda,\alpha,\beta,u,v,\gamma)+k_{24}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}$$
$$\left.\left(\alpha\left\{|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}\right].$$

Proof. By using Lemma 2 and Hölder's integral inequality, we have

$$\begin{aligned}
|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| &\leq u\eta(v,u)\{u+\eta(v,u)\}\Bigg[\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt \\
&\quad+\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\Bigg] \\
&\leq u\eta(v,u)\{u+\eta(v,u)\}\left(\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|^{\gamma}}{(\bar{A}_t)^{2\gamma}}dt\right)^{\frac{1}{\gamma}} \\
&\quad\left(\int_0^{1-\alpha}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}} \\
&\quad+\left(\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|^{\gamma}}{(\bar{A}_t)^{2\gamma}}dt\right)^{\frac{1}{\gamma}} \\
&\quad\left(\int_{1-\alpha}^{1}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}}
\end{aligned}\tag{3.17}$$

(a) (i) If $(\alpha\lambda)^{\frac{1}{\beta}}\leq 1-\alpha$, then

$$\begin{aligned}
&\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|^{\gamma}}{(\bar{A}_t)^{2\gamma}}dt \\
&= \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{(-t^{\beta}+\alpha\lambda)^{\gamma}}{(\bar{A}_t)^{2\gamma}}dt+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{(t^{\beta}-\alpha\lambda)^{\gamma}}{(\bar{A}_t)^{2\gamma}}dt
\end{aligned}$$

$$= \quad k_{19}(\lambda,\alpha,\beta,u,v,\gamma) + k_{20}(\lambda,\alpha,\beta,u,v,\gamma). \qquad (3.18)$$

where

$$k_{19}(\lambda,\alpha,\beta,u,v,\gamma) := \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{(-t^\beta+\alpha\lambda)^\gamma}{((1-t)u+t(u+\eta(v,u)))^{2\gamma}}dt$$

$$k_{20}(\lambda,\alpha,\beta,u,v,\gamma) := \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{(t^\beta-\alpha\lambda)^\gamma}{((1-t)u+t(u+\eta(v,u)))^{2\gamma}}dt$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\int_0^{1-\alpha} \frac{|t^\beta-\alpha\lambda|^\gamma}{(\bar{A}_t)^{2\gamma}}dt = \int_0^{1-\alpha} \frac{(-t^\beta+\alpha\lambda)^\gamma}{(\bar{A}_t)^{2\gamma}}dt$$
$$= k_{21}(\lambda,\alpha,\beta,u,v,\gamma). \qquad (3.19)$$

where

$$k_{21}(\lambda,\alpha,\beta,u,v,\gamma) := \int_0^{1-\alpha} \frac{(-t^\beta+\alpha\lambda)^\gamma}{((1-t)u+t(u+\eta(v,u)))^{2\gamma}}dt$$

(b) (i) If $1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\int_{1-\alpha}^{1} \frac{|(t-1)^\beta-\lambda(\alpha-1)|^\gamma}{(\bar{A}_t)^{2\gamma}}dt$$
$$= \int_{1-\alpha}^{1} \frac{((t-1)^\beta-\lambda(\alpha-1))^\gamma}{(\bar{A}_t)^{2\gamma}}dt$$
$$= k_{22}(\lambda,\alpha,\beta,u,v,\gamma). \qquad (3.20)$$

where

$$k_{22}(\lambda,\alpha,\beta,u,v,\gamma) := \int_{1-\alpha}^{1} \frac{((t-1)^\beta-\lambda(\alpha-1))^\gamma}{((1-t)u+t(u+\eta(v,u)))^{2\gamma}}dt$$

(ii) If $1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\int_{1-\alpha}^{1} \frac{|(t-1)^\beta-\lambda(\alpha-1)|^\gamma}{(\bar{A}_t)^{2\gamma}}dt$$
$$= \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{(-(t-1)^\beta+\lambda(\alpha-1))^\gamma}{(\bar{A}_t)^{2\gamma}}dt$$
$$+ \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{((t-1)^\beta-\lambda(\alpha-1))^\gamma}{(\bar{A}_t)^{2\gamma}}dt$$
$$= k_{23}(\lambda,\alpha,\beta,u,v,\gamma) + k_{24}(\lambda,\alpha,\beta,u,v,\gamma). \qquad (3.21)$$

where

$$k_{23}(\lambda,\alpha,\beta,u,v,\gamma) := \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{(-(t-1)^\beta+\lambda(\alpha-1))^\gamma}{((1-t)u+t(u+\eta(v,u)))^{2\gamma}}dt$$

$$k_{24}(\lambda,\alpha,\beta,u,v,\gamma) := \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{((t-1)^\beta-\lambda(\alpha-1))^\gamma}{((1-t)u+t(u+\eta(v,u)))^{2\gamma}}dt$$

(c) Consider,

$$\int_0^{1-\alpha}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt \tag{3.22}$$

Setting $x=\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}$, so that $dt=\frac{-u\{u+\eta(v,u)\}}{x^2\eta(v,u)}dx$

For $0\leq t\leq 1-\alpha$, we have $u+\eta(v,u)\leq x\leq \frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}$ and hence (3.22) becomes

$$\begin{aligned}
&= -\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{u+\eta(v,u)}^{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\frac{|f'(x)|^{\mu}}{x^2}dx\\
&= \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}^{u+\eta(v,u)}\frac{|f'(x)|^{\mu}}{x^2}dx\\
&= \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\left(\frac{\{u+\eta(v,u)\}\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}{\{u+\eta(v,u)\}-\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\right)\\
&\quad\left(\frac{\{u+\eta(v,u)\}-\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}{\{u+\eta(v,u)\}\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\right)\int_{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}^{u+\eta(v,u)}\frac{|f'(x)|^{\mu}}{x^2}dx
\end{aligned} \tag{3.23}$$

Using Hermite-Hadamard's inequality for relative harmonic preinvex functions, we have

$$\begin{aligned}
&\int_0^{1-\alpha}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\\
\leq\ & \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\left(\frac{\{u+\eta(v,u)\}-\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}{\{u+\eta(v,u)\}\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\right)\\
&\left[\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(u+\eta(v,u))|^{\mu}\right]\int_0^1 h(t)dt\\
=\ & \frac{\bar{A}_{1-\alpha}-u}{\eta(v,u)}\left[\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(u+\eta(v,u))|^{\mu}\right]\int_0^1 h(t)dt\\
=\ & \frac{\alpha u+(1-\alpha)(u+\eta(v,u))-u}{\eta(v,u)}\\
&\left[\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(u+\eta(v,u))|^{\mu}\right]\int_0^1 h(t)dt\\
\leq\ & (1-\alpha)\left[\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(v)|^{\mu}\right]\int_0^1 h(t)dt
\end{aligned} \tag{3.24}$$

Above Inequality holds for $\alpha=1$.

(d) Consider,

$$\int_{1-\alpha}^{1}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt \tag{3.25}$$

Setting $x=\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}$, so that $dt=\frac{-u\{u+\eta(v,u)\}}{x^2\eta(v,u)}dx$

For $1-\alpha\leq t\leq 1$, we have $\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\leq x\leq u$ and hence (3.25) becomes

$$\begin{aligned}
&= -\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}^{u}\frac{|f'(x)|^{\mu}}{x^2}dx\\
&= \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{u}^{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\frac{|f'(x)|^{\mu}}{x^2}dx\\
&= \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\left(\frac{\frac{\{u^2+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}-u}\right)
\end{aligned}$$

$$\left(\frac{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}-u}{\frac{u^2\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\right)\int_u^{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\frac{|f'(x)|^\mu}{x^2}dx \tag{3.26}$$

Using Hermite-Hadamard's inequality for relative harmonic preinvex functions, we have

$$\begin{aligned}
&\int_{1-\alpha}^{1}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dx \\
\leq\ & \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\left(\frac{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}-u}{\frac{u^2\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\right)\left[|f'(u)|^\mu+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu\right]\int_0^1 h(t)dt \\
=\ & \frac{\{u+\eta(v,u)\}-\bar{A}_{1-\alpha}}{\eta(v,u)}\left[|f'(u)|^\mu+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu\right]\int_0^1 h(t)dt \\
=\ & \frac{\{u+\eta(v,u)\}-\alpha u-(1-\alpha)(u+\eta(v,u))}{\eta(v,u)} \\
& \left[|f'(u)|^\mu+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu\right]\int_0^1 h(t)dt \\
\leq\ & \alpha\left[|f'(u)|^\mu+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu\right]\int_0^1 h(t)dt
\end{aligned} \tag{3.27}$$

Above Inequality holds for $\alpha = 0$.

By substituting (3.18) to (3.21), (3.24) and (3.27) in equation (3.17) gives the required result. □

Corollary 5 *Assuming that $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[u, u+\eta(v,u)]$ for $u, u+\eta(v,u) \in M$ with $u < u+\eta(v,u)$, $\beta = 1$ and $\lambda, \alpha \in [0,1]$. If $|f'|^\mu$ is relative harmonically preinvex on M for $\mu > 1$, we have*

(a) *If* $\alpha\lambda \leq 1-\alpha \leq 1+\lambda(\alpha-1)$, *then*

$$\begin{aligned}
|\Psi_f(\lambda,\alpha,1,u,v+\eta(v,u))\ \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\Bigg[(l_{19}(\lambda,\alpha,1,u,v,\gamma)+l_{20}(\lambda,\alpha,1,u,v,\gamma))^{\frac{1}{\gamma}} \\
& \left((1-\alpha)\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu+|f'(v)|^\mu\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}} \\
& +(l_{23}(\lambda,\alpha,1,u,v,\gamma)+l_{24}(\lambda,\alpha,1,u,v,\gamma))^{\frac{1}{\gamma}} \\
& \left(\alpha\left\{|f'(u)|^\mu+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}\Bigg].
\end{aligned}$$

(b) *If* $\alpha\lambda \leq 1+\lambda(\alpha-1) \leq 1-\alpha$, *then*

$$\begin{aligned}
|\Psi_f(\lambda,\alpha,u,u+\eta(v,u))|\ \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\Bigg[(l_{19}(\lambda,\alpha,1,u,v,\gamma)+l_{20}(\lambda,\alpha,1,u,v,\gamma))^{\frac{1}{\gamma}} \\
& \left((1-\alpha)\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu+|f'(v)|^\mu\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}} \\
& +(l_{22}(\lambda,\alpha,1,u,v,\gamma))^{\frac{1}{\gamma}} \\
& \left(\alpha\left\{|f'(u)|^\mu+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}\Bigg].
\end{aligned}$$

(c) *If* $1-\alpha \leq \alpha\lambda \leq 1+\lambda(\alpha-1)$, *then*

$$|\Psi_f(\lambda,\alpha,u,u+\eta(v,u))|\ \leq\ u\eta(v,u)\{u+\eta(v,u)\}\Bigg[(l_{21}(\lambda,\alpha,1,u,v,\gamma))^{\frac{1}{\gamma}}$$

$$\left((1-\alpha)\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(v)|^{\mu}\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}$$
$$+(l_{23}(\lambda,\alpha,1,u,v,\gamma)+l_{24}(\lambda,\alpha,1,u,v,\gamma))^{\frac{1}{\gamma}}$$
$$\left(\alpha\left\{|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}\int_0^1 h(t)dt\right)^{\frac{1}{\mu}}\Bigg].$$

where

$$l_{19}(\lambda,\alpha,1,u,v,\gamma) := \frac{u^{-2\gamma}\alpha\lambda^{1+\gamma}{}_2F_1[1,2\gamma,2+\gamma,-\frac{c\alpha\lambda}{u}]}{1+\gamma}$$

$$l_{20}(\lambda,\alpha,1,u,v,\gamma) := \frac{(u+c-c\alpha)^{1-2\gamma}(1-\alpha-\alpha\lambda)^{1+\gamma}\Gamma(1+\gamma){}_2\tilde{F}_1[1,2-\gamma,2+\gamma,\frac{c(-1+\alpha+\alpha\lambda)}{u+c\alpha\lambda}]}{u+c\alpha\lambda}$$

$$l_{21}(\lambda,\alpha,1,u,v,\gamma) := \frac{u^{1-2\gamma}\alpha\lambda^{1+\gamma}{}_2F_1[1,2-\gamma,2+\gamma,\frac{c\alpha\lambda}{u+c\alpha\lambda}]}{(1+\gamma)(u+c\alpha\lambda)} - \frac{(u+c-c\alpha)^{1-2\gamma}(-1+\alpha+\alpha\lambda)^{1+\gamma}{}_2F_1[1,2-\gamma,2+\gamma,\frac{c(-1+\alpha+\alpha\lambda)}{u+c\alpha\lambda}]}{(1+\gamma)(u+c\alpha\lambda)}$$

$$l_{22}(\lambda,\alpha,1,u,v,\gamma) := -\frac{(u+c)^{1-2\gamma}(-1+\alpha)\lambda(\lambda-\alpha\lambda)^{\gamma}{}_2F_1[1,2-\gamma,2+\gamma,\frac{c(-1+\alpha)\lambda}{u+c(1+(-1+\alpha)\lambda)}]}{(1+\gamma)(u+c(1+(-1+\alpha)\lambda))} - \frac{(u+c-c\alpha)^{1-2\gamma}(\lambda-\alpha(1+\lambda))^{1+\gamma}{}_2F_1[1,2-\gamma,2+\gamma,\frac{c(\alpha-\lambda+\alpha\lambda)}{(u+c(1+(-1+\alpha)\lambda))}]}{(1+\gamma)(u+c(1+(-1+\alpha)\lambda))}$$

$$l_{23}(\lambda,\alpha,1,u,v,\gamma) := \frac{(u+c-c\alpha)^{1-2\gamma}(\alpha+(-1+\alpha)\lambda)^{1+\gamma}{}_2F_1[1,2-\gamma,2+\gamma,\frac{c(\alpha+(-1+\alpha)\lambda)}{u+c+c(-1+\alpha)\lambda}]}{(1+\gamma)(u+c+c\lambda(-1+\alpha))}$$

$$l_{24}(\lambda,\alpha,1,u,v,\gamma) := -\frac{(c+u)^{1-2\gamma}(\lambda(-1+\alpha))^{1+\gamma}{}_2F_1[1,2-\gamma,2+\gamma,\frac{c(-1+\alpha)\lambda}{u+c+c(-1+\alpha)\lambda}]}{(1+\gamma)(u+c+c(-1+\alpha)\lambda)};$$

where $c=\eta(v,u)$.

Remark 11 *If* $\beta=1$, $h(t)=t$ *and* $\eta(v,u)=v-u$, *then the Theorem* 5 *reduces to the Theorem* 2. *This shows that the class of relative harmonic preinvex functions becomes the class of harmonic convex functions.*

Corollary 6 *Assuming that* $f: M=[u,u+\eta(v,u)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f'\in L_1[u,u+\eta(v,u)]$ *for* $u,u+\eta(v,u)\in M$ *with* $u<u+\eta(v,u)$. *If* $|f'|^{\mu}$ *is relative harmonically preinvex on* M *for* $\mu>1$, *then*

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma}}$$
$$[\{s_7(\lambda,u,v,h,\mu)|f'(u)|^{\mu}+s_8(\lambda,u,v,h,\mu)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}$$
$$+\{s_9(\lambda,u,v,h,\mu)|f'(u)|^{\mu}+s_{10}(\lambda,u,v,h,\mu)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].$$

where

$$s_7(\lambda,u,v,h,\mu) := \int_0^{\frac{1}{2}} \frac{h(t)}{(u(1-t)+(u+c)t)^{2\mu}}dt$$

$$s_8(\lambda,u,v,h,\mu) := \int_0^{\frac{1}{2}} \frac{h(1-t)}{(u(1-t)+(u+c)t)^{2\mu}}dt$$

$$s_9(\lambda,u,v,h,\mu) := \int_{\frac{1}{2}}^{1} \frac{h(t)}{(u(1-t)+(u+c)t)^{2\mu}}dt$$

$$s_{10}(\lambda,u,v,h,\mu) := \int_{\frac{1}{2}}^{1} \frac{h(1-t)}{(u(1-t)+(u+c)t)^{2\mu}}dt$$

Proof. From (3.7), we have

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\frac{|(t-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right]$$

Since $|f'|^\mu$ be relative harmonically preinvex on the interval $[u, u+\eta(v,u)]$ with respect to an arbitrary nonnegative function h and for $\mu \in (1,\infty)$, as $t \in [0,1]$ and

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^\mu \leq h(t)|f'(u)|^\mu + h(1-t)|f'(v)|^\mu$$

Using Hölder's integral inequality, we have

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left(\int_0^{\frac{1}{2}}t^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{1}{(\bar{A}_t)^{2\mu}}(h(t)|f'(u)|^\mu + h(1-t)|f'(v)|^\mu)dt\right)^{\frac{1}{\mu}}$$
$$+\left(\int_{\frac{1}{2}}^{1}(1-t)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^{1}\frac{1}{(\bar{A}_t)^{2\mu}}(h(t)|f'(u)|^\mu + h(1-t)|f'(v)|^\mu)dt\right)^{\frac{1}{\mu}}$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left(\int_0^{\frac{1}{2}}t^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{h(t)}{(\bar{A}_t)^{2\mu}}dt|f'(u)|^\mu + \int_0^{\frac{1}{2}}\frac{h(1-t)}{(\bar{A}_t)^{2\mu}}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}$$
$$+\left(\int_{\frac{1}{2}}^{1}(1-t)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^{1}\frac{h(t)}{(\bar{A}_t)^{2\mu}}dt|f'(u)|^\mu + \int_{\frac{1}{2}}^{1}\frac{h(1-t)}{(\bar{A}_t)^{2\mu}}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma}}[\{s_7(\lambda,u,v,h,\mu)|f'(u)|^\mu + s_8(\lambda,u,v,h,\mu)|f'(v)|^\mu\}^{\frac{1}{\mu}}$$
$$+\{s_9(\lambda,u,v,h,\mu)|f'(u)|^\mu + s_{10}(\lambda,u,v,h,\mu)|f'(v)|^\mu\}^{\frac{1}{\mu}}];\ where\ \ c=\eta(v,u).$$

□

Theorem 6 *Assuming that $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[u, u+\eta(v,u)]$ for $u, u+\eta(v,u) \in M$ with $u < u+\eta(v,u), \beta \in (0,1]$ and $\lambda, \alpha \in [0,1]$. If $|f'|$ is relative harmonically preinvex on M, we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \;\leq\; & u\eta(v,u)\{u+\eta(v,u)\}[\{\{k_7(\lambda,\alpha,\beta,u,v,h)+k_8(\lambda,\alpha,\beta,u,v,h)\} \\ & +\{k_{15}(\lambda,\alpha,\beta,u,v,h)+k_{16}(\lambda,\alpha,\beta,u,v,h)\}\}|f'(u)| \\ & +\{\{k_9(\lambda,\alpha,\beta,u,v,h)+k_{10}(\lambda,\alpha,\beta,u,v,h)\} \\ & +\{k_{17}(\lambda,\alpha,\beta,u,v,h)+k_{18}(\lambda,\alpha,\beta,u,v,h)\}\}|f'(v)|].\end{aligned}$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\beta,\alpha,u,u+\eta(v,u))| \;\leq\; & u\eta(v,u)\{u+\eta(v,u)\}[\{\{k_7(\lambda,\alpha,\beta,u,v,h)+k_8(\lambda,\alpha,\beta,u,v,h)\} \\ & +k_{13}(\lambda,\alpha,\beta,u,v,h)\}|f'(u)|+\{\{k_9(\lambda,\alpha,\beta,u,v,h) \\ & +k_{10}(\lambda,\alpha,\beta,u,v,h)\}+k_{14}(\lambda,\alpha,\beta,u,v,h)\}|f'(v)|].\end{aligned}$$

(c) *If* $1-\alpha \leq (\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\beta,\alpha,u,u+\eta(v,u))| \;\leq\; & u\eta(v,u)\{u+\eta(v,u)\}[\{\{k_{15}(\lambda,\alpha,\beta,u,v,h)+k_{16}(\lambda,\alpha,\beta,u,v,h)\} \\ & +k_{11}(\lambda,\alpha,\beta,u,v,h)\}|f'(u)|+\{\{k_{17}(\lambda,\alpha,\beta,u,v,h) \\ & +k_{18}(\lambda,\alpha,\beta,u,v,h)\}+k_{12}(\lambda,\alpha,\beta,u,v,h)\}|f'(v)|].\end{aligned}$$

Where the values of $k_7(\lambda,\alpha,\beta,u,v,h)$ *to* $k_{18}(\lambda,\alpha,\beta,u,v,h)$ *are defined in Theorem* 4.

Proof. By using Lemma 2, we have

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \;\leq\; & u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right. \\ & \left.+\int_{1-\alpha}^{1}\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right].\end{aligned}$$

Since $|f'|$ be relative harmonically preinvex on the interval $[u,u+\eta(v,u)]$ with respect to an arbitrary nonnegative function h and $t\in[0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right| \leq h(t)|f'(u)|+h(1-t)|f'(v)|$$

$$\begin{aligned}& |\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \\ \leq\; & u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}[h(t)|f'(u)|+h(1-t)|f'(v)|]dt\right. \\ & \left.+\int_{1-\alpha}^{1}\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}[h(t)|f'(u)|+h(1-t)|f'(v)|]dt\right] \\ \leq\; & u\eta(v,u)\{u+\eta(v,u)\}\left[\left\{\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}h(t)dt|f'(u)|\right.\right. \\ & \left.+\int_{1-\alpha}^{1}\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}h(t)dt|f'(v)|\right\}+\left\{\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}h(1-t)dt|f'(u)|\right. \\ & \left.\left.+\int_{1-\alpha}^{1}\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}h(1-t)dt|f'(v)|\right\}\right] \qquad (3.28)\end{aligned}$$

(a) (i) If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\begin{aligned}
&\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} h(t)dt \\
= &\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} h(t)dt + \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta - \alpha\lambda}{(\bar{A}_t)^2} h(t)dt \\
= &\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta - \alpha\lambda)}{((1-t)u + t(u+\eta(v,u)))^2} h(t)dt + \\
&\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta - \alpha\lambda}{((1-t)u + t(u+\eta(v,u)))^2} h(t)dt \\
= &k_7(\lambda,\alpha,\beta,u,v,h) + k_8(\lambda,\alpha,\beta,u,v,h). \qquad (3.29)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} h(1-t)dt \\
= &\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} h(1-t)dt + \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta - \alpha\lambda}{(\bar{A}_t)^2} h(1-t)dt \\
= &\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta - \alpha\lambda)}{((1-t)u + t(u+\eta(v,u)))^2} h(1-t)dt \\
&+ \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta - \alpha\lambda}{((1-t)u + t(u+\eta(v,u)))^2} h(1-t)dt \\
= &k_9(\lambda,\alpha,\beta,u,v,h) + k_{10}(\lambda,\alpha,\beta,u,v,h). \qquad (3.30)
\end{aligned}$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\begin{aligned}
&\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} h(t)dt \\
= &\int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} h(t)dt \\
= &\int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{((1-t)u + t(u+\eta(v,u)))^2} h(t)dt \\
= &k_{11}(\lambda,\alpha,\beta,u,v,h). \qquad (3.31)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} h(1-t)dt \\
= &\int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} h(1-t)dt \\
= &\int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{((1-t)u + t(u+\eta(v,u)))^2} h(1-t)dt \\
= &k_{12}(\lambda,\alpha,\beta,u,v,h). \qquad (3.32)
\end{aligned}$$

(b) (i) If $1 + (\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^{1} \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2} h(t)dt \\
= &\int_{1-\alpha}^{1} \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2} h(t)dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_{1-\alpha}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{((1-t)u+t(u+\eta(v,u)))^2} h(t)dt \\
&= k_{13}(\lambda,\alpha,\beta,u,v,h). \qquad (3.33)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{1-\alpha}^{1} \frac{|(t-1)^{\beta}+\lambda(1-\alpha)|}{(\bar{A}_t)^2} h(1-t)dt \\
&= \int_{1-\alpha}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2} h(1-t)dt \\
&= \int_{1-\alpha}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{((1-t)u+t(u+\eta(v,u)))^2} h(1-t)dt \\
&= k_{14}(\lambda,\alpha,\beta,u,v,h). \qquad (3.34)
\end{aligned}
$$

(ii) If $1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$
\begin{aligned}
& \int_{1-\alpha}^{1} \frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2} h(t)dt \\
&= \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2} h(t)dt \\
&\quad + \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2} h(t)dt \\
&= \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{((1-t)u+t(u+\eta(v,u)))^2} h(t)dt \\
&\quad + \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{((1-t)u+t(u+\eta(v,u)))^2} h(t)dt \\
&= k_{15}(\lambda,\alpha,\beta,u,v,h) + k_{16}(\lambda,\alpha,\beta,u,v,h). \qquad (3.35)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{1-\alpha}^{1} \frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2} h(1-t)dt \\
&= \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2} h(1-t)dt \\
&\quad + \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2} h(1-t)dt \\
&= \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{((1-t)u+t(u+\eta(v,u)))^2} h(1-t)dt \\
&\quad + \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{((1-t)u+t(u+\eta(v,u)))^2} h(1-t)dt \\
&= k_{17}(\lambda,\alpha,\beta,u,v,h) + k_{18}(\lambda,\alpha,\beta,u,v,h). \qquad (3.36)
\end{aligned}
$$

By substituting (3.29) to (3.36) in equation (3.28) gives the required result. □

Corollary 7 *Assuming that* $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f' \in L_1[u, u+\eta(v,u)]$ *for* $u, u+\eta(v,u) \in M$ *with* $u < u+\eta(v,u)$, $\beta = 1$ *and* $\lambda, \alpha \in [0,1]$. *If* $|f'|$ *is relative harmonically preinvex on* M, *we have*

(a) *If* $\alpha\lambda \leq 1-\alpha \leq 1+\lambda(\alpha-1)$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,1,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}[\{\{l_7(\lambda,\alpha,1,u,v,h)+l_8(\lambda,\alpha,1,u,v,h)\}\\ &+\{l_{15}(\lambda,\alpha,1,u,v,h)+l_{16}(\lambda,\alpha,1,u,v,h)\}\}|f'(u)|\\ &+\{\{l_9(\lambda,\alpha,1,u,v,h)+l_{10}(\lambda,\alpha,1,u,v,h)\}\\ &+\{l_{17}(\lambda,\alpha,1,u,v,h)+l_{18}(\lambda,\alpha,1,u,v,h)\}\}|f'(v)|].\end{aligned}$$

(b) *If* $\alpha\lambda \leq 1+\lambda(\alpha-1) \leq 1-\alpha$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,1,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}[\{\{l_7(\lambda,\alpha,1,u,v,h)+l_8(\lambda,\alpha,1,u,v,h)\}\\ &+l_{13}(\lambda,\alpha,1,u,v,h)\}|f'(u)|+\{\{l_9(\lambda,\alpha,1,u,v,h)\\ &+l_{10}(\lambda,\alpha,1,u,v,h)\}+l_{14}(\lambda,\alpha,1,u,v,h)\}|f'(v)|].\end{aligned}$$

(c) *If* $1-\alpha \leq \alpha\lambda \leq 1+\lambda(\alpha-1)$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,1,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}[\{\{l_{15}(\lambda,\alpha,1,u,v,h)+l_{16}(\lambda,\alpha,1,u,v,h)\}\\ &+l_{11}(\lambda,\alpha,1,u,v,h)\}|f'(u)|+\{\{l_{17}(\lambda,\alpha,1,u,v,h)\\ &+l_{18}(\lambda,\alpha,1,u,v,h)\}+l_{12}(\lambda,\alpha,1,u,v,h)\}|f'(v)|].\end{aligned}$$

Also, $l_7(\lambda,\alpha,1,u,v,h)$ *and* $l_{18}(\lambda,\alpha,1,u,v,h)$ *are defined in Corollary* 3.

Remark 12 *If* $\beta = 1$, $h(t) = t$ *and* $\eta(v,u) = v-u$, *then the Theorem* 6 *reduces to the Theorem* 3. *This shows that the class of relative harmonic preinvex functions becomes the class of harmonic convex functions.*

s-Harmonic Preinvex Functions

Theorem 7 *Assuming that* $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f' \in L_1[u, u+\eta(v,u)]$ *for* $u, u+\eta(v,u) \in M$ *with* $u < u+\eta(v,u)$, $\beta \in (0,1]$ *and* $\lambda, \alpha \in [0,1]$. *If* $|f'|^\mu$ *is s-harmonic preinvex on* M *for* $\mu > 1$ *with* $\frac{1}{\gamma}+\frac{1}{\mu} = 1$, *we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}[(k_1(\lambda,\alpha,\beta,u,v)+k_2(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\\ &\{(\hat{k_7}(\lambda,\alpha,\beta,u,v,s)+\hat{k_8}(\lambda,\alpha,\beta,u,v,s))|f'(u)|^\mu\\ &+(\hat{k_9}(\lambda,\alpha,\beta,u,v,s)+\hat{k_{10}}(\lambda,\alpha,\beta,u,v,s))|f'(v)|^\mu\}^{\frac{1}{\mu}}\\ &+(k_5(\lambda,\alpha,\beta,u,v)+k_6(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(\hat{k_{15}}(\lambda,\alpha,\beta,u,v,s)\\ &+\hat{k_{16}}(\lambda,\alpha,\beta,u,v,s))|f'(u)|^\mu+(\hat{k_{17}}(\lambda,\alpha,\beta,u,v,s)\\ &+\hat{k_{18}}(\lambda,\alpha,\beta,u,v,s))|f'(v)|^\mu\}^{\frac{1}{\mu}}].\end{aligned}$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1 + (\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}[(k_1(\lambda,\alpha,\beta,u,v)+k_2(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\\ \{(\hat{k_7}(\lambda,\alpha,\beta,u,v,s)+\hat{k_8}(\lambda,\alpha,\beta,u,v,s))|f'(u)|^{\mu}\\ +(\hat{k_9}(\lambda,\alpha,\beta,u,v,s)+\hat{k_{10}}(\lambda,\alpha,\beta,u,v,s))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}\\ +(k_4(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(\hat{k_{13}}(\lambda,\alpha,\beta,u,v,s)|f'(u)|^{\mu}\\ +\hat{k_{14}}(\lambda,\alpha,\beta,u,v,s)|f'(v)|^{\mu})\}^{\frac{1}{\mu}}].\end{aligned}$$

(c) *If* $1-\alpha \leq (\alpha\lambda)^{\frac{1}{\beta}} \leq 1 + (\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}[(k_3(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\\ \{(\hat{k_{11}}(\lambda,\alpha,\beta,u,v,s)|f'(u)|^{\mu}+\hat{k_{12}}(\lambda,\alpha,\beta,u,v,s)|f'(v)|^{\mu})\}^{\frac{1}{\mu}}\\ +(k_5(\lambda,\alpha,\beta,u,v)+k_6(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(\hat{k_{15}}(\lambda,\alpha,\beta,u,v,s)\\ +\hat{k_{16}}(\lambda,\alpha,\beta,u,v,s))|f'(u)|^{\mu}+(\hat{k_{17}}(\lambda,\alpha,\beta,u,v,s)\\ +\hat{k_{18}}(\lambda,\alpha,\beta,u,v,s))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].\end{aligned}$$

Proof. By using Lemma 2 and power mean integral inequality, Similarly to the process of (3.8) to (3.12) and we obtain $k_1,\ k_2,\ k_3,\ k_4,\ k_5$ *and* k_6. Since $|f'|^{\mu}$ be s-harmonic preinvex function on the interval $[u,u+\eta(v,u)]$ for $\mu > 1$ and $s \in (0,1]$, as $t \in [0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^{\mu} \leq t^s|f'(u)|^{\mu}+(1-t)^s|f'(v)|^{\mu}$$

(c) (i) If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\begin{aligned}&\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\\ \leq&\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(\bar{A}_t)^2}[t^s|f'(u)|^{\mu}+(1-t)^s|f'(v)|^{\mu}]dt\\ &+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(\bar{A}_t)^2}[t^s|f'(u)|^{\mu}+(1-t)^s|f'(v)|^{\mu}]dt\\ =&\left[\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(\bar{A}_t)^2}t^s dt+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(\bar{A}_t)^2}t^s dt\right]|f'(u)|^{\mu}\\ &+\left[\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^{\beta}-\alpha\lambda)}{(\bar{A}_t)^2}(1-t)^s dt+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^{\beta}-\alpha\lambda}{(\bar{A}_t)^2}(1-t)^s dt\right]|f'(v)|^{\mu}\\ =&[\hat{k_7}(\lambda,\alpha,\beta,u,v,s)+\hat{k_8}(\lambda,\alpha,\beta,u,v,s)]|f'(u)|^{\mu}\\ &+[\hat{k_9}(\lambda,\alpha,\beta,u,v,s)+\hat{k_{10}}(\lambda,\alpha,\beta,u,v,s)]|f'(v)|^{\mu}. \qquad (3.37)\end{aligned}$$

where

$$\hat{k_7}(\lambda,\alpha,\beta,u,v,s) := \frac{(\alpha\lambda^{\frac{1}{\beta}})^{1+s}}{u^2(u+c\alpha\lambda^{\frac{1}{\beta}})}\left(\frac{\alpha\lambda(u(1+s)-s(u+c\alpha\lambda^{\frac{1}{\beta}})_2F_1[1,1+s,2+s,-\frac{c\alpha\lambda^{\frac{1}{\beta}}}{u}])}{1+s}\right.$$
$$\left.-\frac{(\alpha\lambda^{\frac{1}{\beta}})^{\beta}(u(1+s+\beta)-(u+c\alpha\lambda^{\frac{1}{\beta}})(s+\beta)_2F_1[1,1+s+\beta,2+s+\beta,\frac{-c\alpha\lambda^{\frac{1}{\beta}}}{u}])}{1+s+\beta}\right)$$

$$
\begin{aligned}
\hat{k_8}(\lambda,\alpha,\beta,u,v,s) :=& \frac{-\alpha\lambda}{c^2(1-s)}\Bigg(-\frac{(\frac{1}{1-\alpha})^{-s}(c(-1+s+\alpha-s\alpha)-s(u+c-c\alpha)_2F_1[1,1-s,2-s,\frac{u}{c(-1+\alpha)}])}{(-1+\alpha)(u+c-c\alpha)} \\
&+\frac{(\alpha\lambda^{\frac{-1}{\beta}})^{1-s}(-c(-1+s)\alpha\lambda^{\frac{1}{\beta}}+s(u+c\alpha\lambda^{\frac{1}{\beta}})_2F_1[1,1-s,2-s,\frac{-u\alpha\lambda^{\frac{-1}{\beta}}}{c})])}{u+c\alpha\lambda^{\frac{1}{\beta}}}\Bigg) \\
&-\frac{1}{c^2(-1+s+\beta)}\Bigg(-\frac{(\frac{1}{1-\alpha})^{-s-\beta}(-c(-1+s+\beta))}{(u+c-c\alpha)} \\
&+\frac{(\frac{1}{1-\alpha})^{-s-\beta}(s+\beta)_2F_1[1,1-s-\beta,2-s-\beta,\frac{u}{c(-1+\alpha)}])}{(-1+\alpha)} \\
&+\frac{(\alpha\lambda^{\frac{-1}{\beta}})^{1-s-\beta}(-c\alpha\lambda^{\frac{1}{\beta}}(-1+s+\beta))}{u+c\alpha\lambda^{\frac{1}{\beta}}}+(\alpha\lambda^{\frac{-1}{\beta}})^{1-s-\beta}(s+\beta) \\
&\times{}_2F_1[1,1-s-\beta,2-s-\beta,-\frac{u\alpha\lambda^{\frac{-1}{\beta}}}{c}]\Bigg) \\
\hat{k_9}(\lambda,\alpha,\beta,u,v,s) :=& \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^\beta-\alpha\lambda)}{(u(1-t)+(u+c)t)^2}(1-t)^s dt \\
\hat{k_{10}}(\lambda,\alpha,\beta,u,v,s) :=& \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^\beta-\alpha\lambda}{(u(1-t)+(u+c)t)^2}(1-t)^s dt
\end{aligned}
$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$
\begin{aligned}
&\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt \\
\leq\; &\int_0^{1-\alpha}\frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}[t^s|f'(u)|^\mu+(1-t)^s|f'(v)|^\mu]dt \\
=\; &\int_0^{1-\alpha}\frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}t^s dt|f'(u)|^\mu+\int_0^{1-\alpha}\frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}(1-t)^s dt|f'(v)|^\mu \\
=\; &\hat{k_{11}}(\lambda,\alpha,\beta,u,v,s)|f'(u)|^\mu+\hat{k_{12}}(\lambda,\alpha,\beta,u,v,s)|f'(v)|^\mu. \qquad (3.38)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{k_{11}}(\lambda,\alpha,\beta,u,v,s) :=& \frac{(1-\alpha)^{1+s}}{u^2(u+c-c\alpha)}\Bigg(\frac{(\alpha\lambda(u(1+s)-s(u+c-c\alpha)_2F_1[1,1+s,2+s,\frac{c(-1+\alpha)}{u}]))}{1+s} \\
&-\frac{(1-\alpha)^\beta(u(1+s+\beta)-(u+c-c\alpha)(s+\beta)_2F_1[1,1+s+\beta,2+s+\beta,\frac{c(-1+\alpha)}{u}])}{1+s+\beta}\Bigg) \\
\hat{k_{12}}(\lambda,\alpha,\beta,u,v,s) :=& \int_0^{1-\alpha}\frac{-(t^\beta-\alpha\lambda)}{(u(1-t)+(u+c)t)^2}(1-t)^s dt
\end{aligned}
$$

(d) (i) If $1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$
\begin{aligned}
&\int_{1-\alpha}^{1}\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt \\
\leq\; &\int_{1-\alpha}^{1}\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}[t^s|f'(u)|^\mu+(1-t)^s|f'(v)|^\mu]dt \\
=\; &\int_{1-\alpha}^{1}\frac{(t-1)^\beta-\lambda(\alpha-1)}{(\bar{A}_t)^2}t^s dt|f'(u)|^\mu \\
&+\int_{1-\alpha}^{1}\frac{(t-1)^\beta-\lambda(\alpha-1)}{(\bar{A}_t)^2}(1-t)^s dt|f'(v)|^\mu \\
=\; &\hat{k_{13}}(\lambda,\alpha,\beta,u,v,s)|f'(u)|^\mu+\hat{k_{14}}(\lambda,\alpha,\beta,u,v,s)|f'(v)|^\mu. \qquad (3.39)
\end{aligned}
$$

where

$$\hat{k_{13}}(\lambda,\alpha,\beta,u,v,s) := -\frac{(-1)^{\beta}(1-\alpha)^{s}F_1[1+s,-\beta,2,2+s,1-\alpha,\frac{c(-1+\alpha)}{u}]}{u^2(1+s)}$$
$$+\frac{(-1)^{\beta}(1-\alpha)^{s}\alpha F_1[1+s,-\beta,2,2+s,1-\alpha,\frac{c(-1+\alpha)}{u}]}{u^2(1+s)}$$
$$-\frac{1}{(c^2(u+c)(-1+s)(-1+\alpha)(u+c-c\alpha))}\lambda(-c(-1+s)(-1+\alpha)$$
$$\times(u(-1+(1-\alpha)^s)+c(-1+(1-\alpha)^s+\alpha))-(u+c)s(-1+\alpha)$$
$$\times(u+c-c\alpha)_2F_1[1,1-s,2-s,-\frac{u}{c}]-(u+c)s(1-\alpha)^s(u+c-c\alpha)$$
$$\times{}_2F_1[1,1-s,2-s,\frac{u}{c(-1+\alpha)}])$$
$$+\frac{1}{(c^2(u+c)(-1+s)(-1+\alpha)(u+c-c\alpha))}\alpha\lambda(-c(-1+s)(-1+\alpha)$$
$$\times(u(-1+(1-\alpha)^s)+c(-1+(1-\alpha)^s+\alpha))-(u+c)s(-1+\alpha)$$
$$\times(u+c-c\alpha)_2F_1[1,1-s,2-s,-\frac{u}{c}]-(u+c)s(1-\alpha)^s(u+c-c\alpha)$$
$$\times{}_2F_1[1,1-s,2-s,\frac{u}{c(-1+\alpha)}])$$
$$+\frac{(-1)^{\beta}\Gamma(1+s)\Gamma(1+\beta)_2F_1[2,1+s,2+s+\beta,-\frac{c}{u}]}{u^2\Gamma(2+s+\beta)}$$

$$\hat{k_{14}}(\lambda,\alpha,\beta,u,v,s) := \int_{1-\alpha}^{1}\frac{(t-1)^{\beta}+\lambda(1-\alpha)}{((1-t)u+t(u+c))^2}(1-t)^s dt$$

(ii) If $1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt$$
$$\leq \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2}[t^s|f'(u)|^{\mu}+(1-t)^s|f'(v)|^{\mu}]dt$$
$$+\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}[t^s|f'(u)|^{\mu}+(1-t)^s|f'(v)|^{\mu}]dt$$
$$= \left[\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2}t^s dt\right.$$
$$\left.+\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}t^s dt\right]|f'(u)|^{\mu}$$
$$+\left[\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2}(1-t)^s dt\right.$$
$$\left.+\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}(1-t)^s dt\right]|f'(v)|^{\mu}$$
$$= [\hat{k_{15}}(\lambda,\alpha,\beta,u,v,s)+\hat{k_{16}}(\lambda,\alpha,\beta,u,v,s)]|f'(u)|^{\mu}$$
$$+[\hat{k_{17}}(\lambda,\alpha,\beta,u,v,s)+\hat{k_{18}}(\lambda,\alpha,\beta,u,v,s)]|f'(v)|^{\mu}. \quad (3.40)$$

where,

$$\hat{k_{15}}(\lambda,\alpha,\beta,u,v,s) := (\frac{-1}{u^2(1+s)}\alpha^{-\beta}(-((\alpha-1)\lambda)^{\frac{1}{\beta}})^{-\beta}\Big(-(1-\alpha)^{s+1}$$
$$\times(\alpha((-1+\alpha)\lambda)^{\frac{1}{\beta}})^{\beta}F_1[1+s,-\beta,2,2+s,1-\alpha,\frac{c(\alpha-1)}{u}]$$

$$
\begin{aligned}
&+\alpha^{\beta}(((-1+\alpha)\lambda)^{\frac{1}{\beta}})^{\beta}(1+((-1+\alpha)\lambda^{\frac{1}{\beta}})^{1+s} \\
&\times F_1[1+s,-\beta,2,2+s,1+((\alpha-1)\lambda)^{\frac{1}{\beta}},-\frac{c(1+((\alpha-1)\lambda))^{\frac{1}{\beta}}}{u}]\Big) \\
&+\frac{(1-\alpha)}{c^2(s-1)}\lambda\Big(\frac{c(1-s)(1-\alpha)^s}{u+c-c\alpha}+\frac{(c+\frac{u}{1-\alpha})(1-\alpha)^s{}_2F_1[1,1-s,2-s,\frac{u}{c(-1+\alpha)}]}{u+c-c\alpha} \\
&+\left(\frac{1}{1+((-1+\alpha)\lambda)^{\frac{1}{\beta}}}\right)^{1-s}\Big(\frac{c(-1+s)(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})}{u+c+c((-1+\alpha)\lambda)^{\frac{1}{\beta}}} \\
&-\frac{{}_2F_1\left[1,1-s,2-s,-\frac{u}{c(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})}\right]}{u+c+c((-1+\alpha)\lambda)^{\frac{1}{\beta}}}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\hat{k_{16}}(\lambda,\alpha,\beta,u,v,s) &:= \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{((1-t)u+t(u+c))^2}t^s dt \\
\hat{k_{17}}(\lambda,\alpha,\beta,u,v,s) &:= \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{((1-t)u+t(u+c))^2}(1-t)^s dt \\
\hat{k_{18}}(\lambda,\alpha,\beta,u,v,s) &:= \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(1-t)u+t(u+c))^2}(1-t)^s dt
\end{aligned}
$$

Where $c=\eta(v,u)$. By substituting (3.9) to (3.12) and (3.37) to (3.40) in equation (3.8) gives the required result. ◻

Corollary 8 *Assuming that* $f: M=[u,u+\eta(v,u)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f'\in L_1[u,u+\eta(v,u)]$ *for* $u,u+\eta(v,u)\in M$ *with* $u<u+\eta(v,u)$*. If* $|f'|^{\mu}$ *is s-harmonic preinvex on* M *for* $\mu>1$ *with* $\frac{1}{\gamma}+\frac{1}{\mu}=1$*, then*

$$
\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{u}^{u+\eta(v,u)}\frac{f(z)}{z^2}dz-f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|
$$

$$
\begin{aligned}
&\leq u\eta(v,u)\{u+\eta(v,u)\}[s_5^{\frac{1}{\gamma}}(\lambda,u,v)\{\hat{s_1}(\lambda,u,v,s)|f'(u)|^{\mu}+\hat{s_2}(\lambda,u,v,s) \\
&\times|f'(v)|^{\mu}\}^{\frac{1}{\mu}}+s_6^{\frac{1}{\gamma}}(\lambda,u,v)\{\hat{s_3}(\lambda,u,v,s)|f'(u)|^{\mu}+\hat{s_4}(\lambda,u,v,s)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{s_1}(\lambda,u,v,s) &:= \frac{2^{-2-s}\left(\frac{2u}{2u+c}-(1+s)\Gamma(2+s){}_2\tilde{F}_1[1,2+s,3+s,-\frac{c}{2u}]\right)}{u^2} \\
\hat{s_2}(\lambda,u,v,s) &:= \frac{F_1[2,-s,2,3,\frac{1}{2},-\frac{c}{2u}]}{8u^2} \\
\hat{s_3}(\lambda,u,v,s) &:= \frac{u(1+s)-(u+us+cs){}_2F1[1,1+s,2+s,-\frac{c}{u}]}{u^2c(1+s)} \\
&\quad+\frac{2^{-1-s}(-2(u+c))}{uc(2u+c)}+\frac{(u+us+cs){}_2F_1[1,1+s,2+s,-\frac{c}{2u}]}{u^2c(1+s)} \\
\hat{s_4}(\lambda,u,v,s) &:= \int_{\frac{1}{2}}^{1}\frac{-(t-1)(1-t)^s}{((1-t)u+t(u+c))^2}dt;\ where\ c=\eta(v,u).
\end{aligned}
$$

Also, $s_5(\lambda,u,v)$ *and* $s_6(\lambda,u,v)$ *are defined in Corollary* 4.

Proof. From (3.7), we have

$$
\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{u}^{u+\eta(v,u)}\frac{f(z)}{z^2}dz-f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|
$$

$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt + \int_{\frac{1}{2}}^{1}\frac{|t-1|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right].$$

By using power mean integral inequality , we have

$$\begin{aligned}\leq\ & u\eta(v,u)\{u+\eta(v,u)\}\\ & \left[\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right.\\ & \left.+\left(\int_{\frac{1}{2}}^{1}\frac{|t-1|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^{1}\frac{|t-1|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right]\end{aligned}$$

Since $|f'|^{\mu}$ be s-harmonic preinvex function on the interval $[u,u+\eta(v,u)]$ for $\mu>1$ and $s\in(0,1]$, as $t\in[0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^{\mu}\leq t^s|f'(u)|^{\mu}+(1-t)^s|f'(v)|^{\mu}$$

$$\begin{aligned}& \left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz-f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|\\ \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\\ & \left[\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}[t^s|f'(u)|^{\mu}+(1-t)^s|f'(v)|^{\mu}]dt\right)^{\frac{1}{\mu}}\right.\\ & \left.+\left(\int_{\frac{1}{2}}^{1}\frac{|t-1|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^{1}\frac{|t-1|}{(\bar{A}_t)^2}[t^s|f'(u)|^{\mu}+(1-t)^s|f'(v)|^{\mu}]dt\right)^{\frac{1}{\mu}}\right]\\ =\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\left(\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{t^{s+1}}{(\bar{A}_t)^2}dt|f'(u)|^{\mu}\right.\right.\\ & \left.+\int_0^{\frac{1}{2}}\frac{t(1-t)^s}{(\bar{A}_t)^2}dt|f'(v)|^{\mu}\right)^{\frac{1}{\mu}}+\left(\int_{\frac{1}{2}}^{1}\frac{-(t-1)}{(\bar{A}_t)^2}dt\right)^{\frac{1}{\gamma}}\\ & \left.\left(\int_{\frac{1}{2}}^{1}\frac{-(t-1)t^s}{(\bar{A}_t)^2}dt|f'(u)|^{\mu}+\int_{\frac{1}{2}}^{1}\frac{-(t-1)(1-t)^s}{(\bar{A}_t)^2}dt|f'(v)|^{\mu}\right)^{\frac{1}{\mu}}\right].\\ \leq\ & u\eta(v,u)\{u+\eta(v,u)\}[s_5^{\frac{1}{\gamma}}(\lambda,u,v)\{\hat{s_1}(\lambda,u,v,s)|f'(u)|^{\mu}+\hat{s_2}(\lambda,u,v,s)\\ & \times|f'(v)|^{\mu}\}^{\frac{1}{\mu}}+s_6^{\frac{1}{\gamma}}(\lambda,u,v)\{\hat{s_3}(\lambda,u,v,s)|f'(u)|^{\mu}+\hat{s_4}(\lambda,u,v,s)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].\end{aligned}$$

□

Theorem 8 *Assuming that* $f: M=[u,u+\eta(v,u)]\subseteq\mathbb{R}\backslash\{0\}\longrightarrow\mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f'\in L_1[u,u+\eta(v,u)]$ *for* $u,u+\eta(v,u)\in M$ *with* $u<u+\eta(v,u),\beta\in(0,1]$ *and* $\lambda,\alpha\in[0,1]$*. If* $|f'|^{\mu}$ *is s-harmonic preinvex on* M *for* $\mu>1$*, we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}}\leq 1-\alpha\leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$*, then*

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))|\leq u\eta(v,u)\{u+\eta(v,u)\}\Big[(k_{19}(\lambda,\alpha,\beta,u,v,\gamma)+k_{20}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}$$

$$\left((1-\alpha)\frac{\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(v)|^{\mu}\right\}}{s+1}\right)^{\frac{1}{\mu}}+(k_{23}(\lambda,\alpha,\beta,u,v,\gamma)$$

$$+k_{24}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}\left(\alpha\frac{\left\{|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}}{s+1}\right)^{\frac{1}{\mu}}\Bigg].$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \le 1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \le 1-\alpha$, *then*

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \le u\eta(v,u)\{u+\eta(v,u)\}\Bigg[(k_{19}(\lambda,\alpha,\beta,u,v,\gamma)+k_{20}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}$$

$$\left((1-\alpha)\frac{\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(v)|^{\mu}\right\}}{s+1}\right)^{\frac{1}{\mu}}+(k_{22}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}$$

$$\left(\alpha\frac{\left\{|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}}{s+1}\right)^{\frac{1}{\mu}}\Bigg].$$

(c) *If* $1-\alpha \le (\alpha\lambda)^{\frac{1}{\beta}} \le 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \le u\eta(v,u)\{u+\eta(v,u)\}\Bigg[(k_{21}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}$$

$$\left((1-\alpha)\frac{\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(v)|^{\mu}\right\}}{s+1}\right)^{\frac{1}{\mu}}+(k_{23}(\lambda,\alpha,\beta,u,v,\gamma)$$

$$+k_{24}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}\left(\alpha\frac{\left\{|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}}{s+1}\right)^{\frac{1}{\mu}}\Bigg].$$

where $k_{19}(\lambda,\alpha,\beta,u,v,\gamma)$ *to* $k_{24}(\lambda,\alpha,\beta,u,v,\gamma)$ *are defined in Theorem* 5.

Proof. By using Lemma 2 and Hölder's integral inequality, we have
Similarly to the process of (3.17) to (3.21) and we obtain $k_{19}, k_{20}, k_{21}, k_{22}, k_{23}$ *and* k_{24}.

(c) Consider,

$$\int_0^{1-\alpha}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt$$

Now similarly to the process of (3.22) to (3.23).

Using Hermite-Hadamard's inequality for s-harmonic preinvex functions, we have

$$\int_0^{1-\alpha}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt$$
$$\le \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\left(\frac{\{u+\eta(v,u)\}-\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}{\{u+\eta(v,u)\}\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\right)\left[\frac{|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}+|f'(u+\eta(v,u))|^{\mu}}{s+1}\right]$$

$$
\begin{aligned}
&= \frac{\bar{A}_{1-\alpha}-u}{\eta(v,u)}\left[\frac{|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}+|f'(u+\eta(v,u))|^{\mu}}{s+1}\right]\\
&= \frac{\alpha u+(1-\alpha)(u+\eta(v,u))-u}{\eta(v,u)}\left[\frac{|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}+|f'(u+\eta(v,u))|^{\mu}}{s+1}\right]\\
&\leq (1-\alpha)\left[\frac{|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}+|f'(v)|^{\mu}}{s+1}\right] \qquad (3.41)
\end{aligned}
$$

Above Inequality holds for $\alpha = 1$.

(d) Consider,

$$\int_{1-\alpha}^{1}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt$$

Similarly to the process of (3.25) to (3.26).

Using Hermite-Hadamard's inequality for s-harmonic preinvex functions, we have

$$
\begin{aligned}
&\int_{1-\alpha}^{1}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\\
&\leq \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\left(\frac{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}-u}{\frac{u^2\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\right)\left[\frac{|f'(u)|^{\mu}+|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}}{s+1}\right]\\
&= \frac{\{u+\eta(v,u)\}-\bar{A}_{1-\alpha}}{\eta(v,u)}\left[\frac{|f'(u)|^{\mu}+|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}}{s+1}\right]\\
&= \frac{\{u+\eta(v,u)\}-\alpha u-(1-\alpha)(u+\eta(v,u))}{\eta(v,u)}\left[\frac{|f'(u)|^{\mu}+|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}}{s+1}\right]dt\\
&\leq \alpha\left[\frac{|f'(u)|^{\mu}+|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}}{s+1}\right] \qquad (3.42)
\end{aligned}
$$

Above Inequality holds for $\alpha = 0$.

By substituting (3.18) to (3.21), (3.41) and (3.42) in equation (3.17) gives the required result. □

Corollary 9 *Assuming that $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[u, u+\eta(v,u)]$ for $u, u+\eta(v,u) \in M$ with $u < u+\eta(v,u)$. If $|f'|^{\mu}$ s-harmonic preinvex on M for $\mu > 1$, then*

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{u}^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$

$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma}}[\{\hat{s_7}(\lambda,u,v,s,\mu)|f'(u)|^{\mu}$$

$$+\hat{s_8}(\lambda,u,v,s,\mu)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}+\{\hat{s_9}(\lambda,u,v,s,\mu)|f'(u)|^{\mu}+\hat{s_{10}}(\lambda,u,v,s,\mu)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].$$

where

$$\hat{s_7}(\lambda,u,v,s,\mu) := \frac{2^{-1-s}u^{-2\mu}{}_2F_1[2\mu,1+s,2+s,-\frac{c}{2u}]}{1+s}$$

$$\hat{s_8}(\lambda,u,v,s,\mu) := -\frac{u^{1-2\mu}{}_2F_1[1,2-2\mu+s,2-2\mu,\frac{u}{u+c}]}{(u+c)(1-2\mu)} + \frac{2^{-2+2\mu-s}(2u+c)^{1-2\mu}{}_2F_1[1,2-2\mu+s,2-2\mu,\frac{2u+c}{2(u+c)}]}{(u+c)(1-2\mu)}$$

$$\hat{s_9}(\lambda,u,v,s,\mu) := \frac{(u+c)^{1-2\mu}{}_2F_1[1,2-2\mu+s,2+s,-\frac{c}{u}]}{u+us} - \frac{2^{-2+2\mu-s}(2u+c)^{1-2\mu}{}_2F_1[1,2-2\mu+s,2+s,-\frac{c}{2u}]}{u+us}$$

$$\hat{s_{10}}(\lambda,u,b,s,\mu) := \frac{(u+\frac{c}{2})^{-2\mu}(\frac{c}{u+c})^{-s}(u+c)^{-2\mu}}{2c(-1+2\mu)\Gamma(2-2\mu+s)}(-2(u+\frac{c}{2})^{2\mu} (u+c)\Gamma(2-2\mu)\Gamma(1+s)+(u+c)^{2\mu}(2u+c) \Gamma(2-2\mu+s){}_2F_1[1-2\mu,-s,2-2\mu,\frac{2u+c}{2(u+c)}]);\ where\ c=\eta(v,u).$$

Proof. From (3.7), we have

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right| \le u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt + \int_{\frac{1}{2}}^1\frac{|(t-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right]$$

Since $|f'|^\mu$ be s-harmonic preinvex function on the interval $[u,u+\eta(v,u)]$ for $\mu>1$ and $s\in(0,1]$, as $t\in[0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^\mu \le t^s|f'(u)|^\mu + (1-t)^s|f'(v)|^\mu$$

Using Hölder integral inequality, we have

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$

$$\le u\eta(v,u)\{u+\eta(v,u)\}\left(\int_0^{\frac{1}{2}}t^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{1}{(\bar{A}_t)^{2\mu}}(t^s|f'(u)|^\mu+(1-t)^s|f'(v)|^\mu)dt\right)^{\frac{1}{\mu}} + \left(\int_{\frac{1}{2}}^1(1-t)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^1\frac{1}{(\bar{A}_t)^{2\mu}}(t^s|f'(u)|^\mu+(1-t)^s|f'(v)|^\mu)dt\right)^{\frac{1}{\mu}}$$

$$\le u\eta(v,u)\{u+\eta(v,u)\}\left(\int_0^{\frac{1}{2}}t^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{t^s}{(\bar{A}_t)^{2\mu}}dt|f'(u)|^\mu+\int_0^{\frac{1}{2}}\frac{(1-t)^s}{(\bar{A}_t)^{2\mu}}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}} + \left(\int_{\frac{1}{2}}^1(1-t)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^1\frac{t^s}{(\bar{A}_t)^{2\mu}}dt|f'(u)|^\mu+\int_{\frac{1}{2}}^1\frac{(1-t)^s}{(\bar{A}_t)^{2\mu}}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}$$

$$\le u\eta(v,u)\{u+\eta(v,u)\}\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma}}[\{\hat{s_7}(\lambda,u,v,s,\mu)|f'(u)|^\mu+\hat{s_8}(\lambda,u,v,s,\mu)|f'(v)|^\mu\}^{\frac{1}{\mu}} + \{\hat{s_9}(\lambda,u,v,s,\mu)|f'(u)|^\mu+\hat{s_{10}}(\lambda,u,v,s,\mu)|f'(v)|^\mu\}^{\frac{1}{\mu}}].$$

□

Theorem 9 *Assuming that* $f: M=[u,u+\eta(v,u)]\subseteq\mathbb{R}\backslash\{0\}\longrightarrow\mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f'\in L_1[u,u+\eta(v,u)]$ *for* $u,u+\eta(v,u)\in M$ *with* $u<u+\eta(v,u)$, $\beta\in(0,1]$ *and* $\lambda,\alpha\in[0,1]$. *If* $|f'|$ *is s-harmonic preinvex on* M, *we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\\ & [\{\{\hat{k_7}(\lambda,\alpha,\beta,u,v,s)+\hat{k_8}(\lambda,\alpha,\beta,u,v,s)\}\\ & +\{\hat{k_{15}}(\lambda,\alpha,\beta,u,v,s)+\hat{k_{16}}(\lambda,\alpha,\beta,u,v,s)\}\}|f'(u)|\\ & +\{\{\hat{k_9}(\lambda,\alpha,\beta,u,v,s)+\hat{k_{10}}(\lambda,\alpha,\beta,u,v,s)\}\\ & +\{\hat{k_{17}}(\lambda,\alpha,\beta,u,v,s)+\hat{k_{18}}(\lambda,\alpha,\beta,u,v,s)\}\}|f'(v)|].\end{aligned}$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\beta,\alpha,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\\ & [\{\{\hat{k_7}(\lambda,\alpha,\beta,u,v,s)+\hat{k_8}(\lambda,\alpha,\beta,u,v,s)\}\\ & +\hat{k_{13}}(\lambda,\alpha,\beta,u,v,s)\}|f'(u)|+\{\{\hat{k_9}(\lambda,\alpha,\beta,u,v,s)\\ & +\hat{k_{10}}(\lambda,\alpha,\beta,u,v,s)\}+\hat{k_{14}}(\lambda,\alpha,\beta,u,v,s)\}|f'(v)|].\end{aligned}$$

(c) *If* $1-\alpha \leq (\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\beta,\alpha,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\\ & [\{\{\hat{k_{15}}(\lambda,\alpha,\beta,u,v,s)+\hat{k_{16}}(\lambda,\alpha,\beta,u,v,s)\}\\ & +\hat{k_{11}}(\lambda,\alpha,\beta,u,v,s)\}|f'(u)|+\{\{\hat{k_{17}}(\lambda,\alpha,\beta,u,v,s)\\ & +\hat{k_{18}}(\lambda,\alpha,\beta,u,v,s)\}+\hat{k_{12}}(\lambda,\alpha,\beta,u,v,s)\}|f'(v)|].\end{aligned}$$

where $\hat{k_7}(\lambda,\alpha,\beta,u,v,s)$ to $\hat{k_{18}}(\lambda,\alpha,\beta,u,v,s)$ are defined in Theorem 4.

Proof. By using Lemma 2, we have

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right.\\ & \left.+\int_{1-\alpha}^1\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right]\end{aligned}$$

Since $|f'|$ be s-harmonic preinvex function on the interval $[u,u+\eta(v,u)]$ and $s\in(0,1]$, as $t\in[0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right| \leq t^s|f'(u)|+(1-t)^s|f'(v)|$$

$$\begin{aligned}&|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))|\\ &\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}[t^s|f'(u)|+(1-t)^s|f'(v)|]dt\right.\\ &\quad\left.+\int_{1-\alpha}^1\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}[t^s|f'(u)|+(1-t)^s|f'(v)|]dt\right]\\ &\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\left\{\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}t^sdt|f'(u)|+\int_{1-\alpha}^1\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}t^sdt|f'(v)|\right\}\right.\\ &\quad\left.+\left\{\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}(1-t)^sdt|f'(u)|+\int_{1-\alpha}^1\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}(1-t)^sdt|f'(v)|\right\}\right] \qquad (3.43)\end{aligned}$$

(a) (i) If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\begin{aligned} & \int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} t^s dt \\ = & \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} t^s dt + \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta - \alpha\lambda}{(\bar{A}_t)^2} t^s dt \\ = & \hat{k_7}(\lambda, \alpha, \beta, u, v, s) + \hat{k_8}(\lambda, \alpha, \beta, u, v, s). \end{aligned} \quad (3.44)$$

and

$$\begin{aligned} & \int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} (1-t)^s dt \\ = & \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} (1-t)^s dt + \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta - \alpha\lambda}{(\bar{A}_t)^2} (1-t)^s dt \\ = & \hat{k_9}(\lambda, \alpha, \beta, u, v, s) + \hat{k_{10}}(\lambda, \alpha, \beta, u, v, s). \end{aligned} \quad (3.45)$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\begin{aligned} & \int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} t^s dt \\ = & \int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} t^s dt \\ = & \hat{k_{11}}(\lambda, \alpha, \beta, u, v, s). \end{aligned} \quad (3.46)$$

and

$$\begin{aligned} & \int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} (1-t)^s dt \\ = & \int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} (1-t)^s dt \\ = & \hat{k_{12}}(\lambda, \alpha, \beta, u, v, s). \end{aligned} \quad (3.47)$$

(b) (i) If $1 + (\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\begin{aligned} & \int_{1-\alpha}^1 \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2} t^s dt \\ = & \int_{1-\alpha}^1 \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2} t^s dt \\ = & \hat{k_{13}}(\lambda, \alpha, \beta, u, v, s). \end{aligned} \quad (3.48)$$

and

$$\begin{aligned} & \int_{1-\alpha}^1 \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2} (1-t)^s dt \\ = & \int_{1-\alpha}^1 \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2} (1-t)^s dt \\ = & \hat{k_{14}}(\lambda, \alpha, \beta, u, v, s). \end{aligned} \quad (3.49)$$

(ii) If $1 + (\lambda(\alpha-1))^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\int_{1-\alpha}^1 \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2} t^s dt$$

$$
\begin{aligned}
&= \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2} t^s dt \\
&\quad + \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2} t^s dt \\
&= \hat{k_{15}}(\lambda,\alpha,\beta,u,v,s) + \hat{k_{16}}(\lambda,\alpha,\beta,u,v,s). \qquad (3.50)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{1-\alpha}^{1} \frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}(1-t)^s dt \\
&= \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2}(1-t)^s dt \\
&\quad + \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}(1-t)^s dt \\
&= \hat{k_{17}}(\lambda,\alpha,\beta,u,v,s) + \hat{k_{18}}(\lambda,\alpha,\beta,u,v,s). \qquad (3.51)
\end{aligned}
$$

By substituting (3.44) to (3.51) in equation (3.43) gives the required result. □

s - Harmonic Godunova - Levin Preinvex Functions

Theorem 10 *Assuming that $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[u, u+\eta(v,u)]$ for $u, u+\eta(v,u) \in M$ with $u < u+\eta(v,u)$, $\beta \in (0,1]$ and $\lambda, \alpha \in [0,1]$. If $|f'|^{\mu}$ is s-harmonic Godunova-Levin preinvex on M for $\mu > 1$ with $\frac{1}{\gamma}+\frac{1}{\mu}=1$, we have*

(a) *If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, then*

$$
\begin{aligned}
|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq{} & u\eta(v,u)\{u+\eta(v,u)\}[(k_1(\lambda,\alpha,\beta,u,v)+k_2(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}} \\
& \{(k_7^*(\lambda,\alpha,\beta,u,v,-s)+k_8^*(\lambda,\alpha,\beta,u,v,-s)) \\
& |f'(u)|^{\mu}+(k_9^*(\lambda,\alpha,\beta,u,v,-s)+k_{10}^*(\lambda,\alpha,\beta,u,v,-s)) \\
& |f'(v)|^{\mu}\}^{\frac{1}{\mu}}+(k_5(\lambda,\alpha,\beta,u,v)+k_6(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}} \\
& \{(k_{15}^*(\lambda,\alpha,\beta,u,v,-s)+k_{16}^*(\lambda,\alpha,\beta,u,v,-s))|f'(u)|^{\mu} \\
& +(k_{17}^*(\lambda,\alpha,\beta,u,v,-s)+k_{18}^*(\lambda,\alpha,\beta,u,v,-s))|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].
\end{aligned}
$$

(b) *If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, then*

$$
\begin{aligned}
|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq{} & u\eta(v,u)\{u+\eta(v,u)\}[(k_1(\lambda,\alpha,\beta,u,v)+k_2(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}} \\
& \{(k_7^*(\lambda,\alpha,\beta,u,v,-s)+k_8^*(\lambda,\alpha,\beta,u,v,-s))|f'(u)|^{\mu} \\
& +(k_9^*(\lambda,\alpha,\beta,u,v,-s)+k_{10}^*(\lambda,\alpha,\beta,u,v,-s))|f'(v)|^{\mu}\}^{\frac{1}{\mu}} \\
& +(k_4(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(k_{13}^*(\lambda,\alpha,\beta,u,v,-s)|f'(u)|^{\mu} \\
& +k_{14}^*(\lambda,\alpha,\beta,u,v,-s)|f'(v)|^{\mu})\}^{\frac{1}{\mu}}].
\end{aligned}
$$

(c) *If* $1-\alpha \le (\alpha\lambda)^{\frac{1}{\beta}} \le 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \le\ & u\eta(v,u)\{u+\eta(v,u)\}[(k_3(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(k_{11}^*(\lambda,\alpha,\beta,u,v,-s)\\ & |f'(u)|^\mu + k_{12}^*(\lambda,\alpha,\beta,u,v,-s)|f'(v)|^\mu)\}^{\frac{1}{\mu}} + (k_5(\lambda,\alpha,\beta,u,v)\\ & +k_6(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(k_{15}^*(\lambda,\alpha,\beta,u,v,-s)+k_{16}^*(\lambda,\alpha,\beta,u,v,-s))\\ & |f'(u)|^\mu+(k_{17}^*(\lambda,\alpha,\beta,u,v,-s)+k_{18}^*(\lambda,\alpha,\beta,u,v,-s))|f'(v)|^\mu\}^{\frac{1}{\mu}}].\end{aligned}$$

Proof. By using Lemma 2 and power mean integral inequality,
Similarly to the process of (3.8) to (3.12) and we obtain $k_1,\ k_2,\ k_3,\ k_4,\ k_5$ *and* k_6. Since $|f'|^\mu$ be s-harmonic Godunova-Levin preinvex function on the interval $[u,u+\eta(v,u)]$ for $\mu>1$ and $s\in(0,1]$, as $t\in[0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^\mu \le t^{-s}|f'(u)|^\mu + (1-t)^{-s}|f'(v)|^\mu$$

(c) (i) If $(\alpha\lambda)^{\frac{1}{\beta}} \le 1-\alpha$, then

$$\begin{aligned}&\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt\\
\le\ & \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}[t^{-s}|f'(u)|^\mu+(1-t)^{-s}|f'(v)|^\mu]dt\\
&+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^\beta-\alpha\lambda}{(\bar{A}_t)^2}[t^{-s}|f'(u)|^\mu+(1-t)^{-s}|f'(v)|^\mu]dt\\
=\ & \left[\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}t^{-s}dt+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^\beta-\alpha\lambda}{(\bar{A}_t)^2}t^{-s}dt\right]|f'(u)|^\mu\\
&+\left[\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}(1-t)^{-s}dt+\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^\beta-\alpha\lambda}{(\bar{A}_t)^2}(1-t)^{-s}dt\right]|f'(v)|^\mu\\
=\ & [k_7^*(\lambda,\alpha,\beta,u,v,-s)+k_8^*(\lambda,\alpha,\beta,u,v,-s)]|f'(u)|^\mu\\
&+[k_9^*(\lambda,\alpha,\beta,u,v,-s)+k_{10}^*(\lambda,\alpha,\beta,u,v,-s)]|f'(v)|^\mu. \qquad (3.52)\end{aligned}$$

where

$$\begin{aligned}k_7^*(\lambda,\alpha,\beta,u,v,-s) :=\ & \frac{(\alpha\lambda^{\frac{1}{\beta}})^{1-s}}{u^2(u+c\alpha\lambda^{\frac{1}{\beta}})}\left(-\frac{\alpha\lambda(u(1-s)+s(u+c\alpha\lambda^{\frac{1}{\beta}})_2F_1[1,1-s,2-s,\frac{-c\alpha\lambda^{\frac{1}{\beta}}}{u}])}{-1+s}\right.\\
&+\frac{(\alpha\lambda^{\frac{1}{\beta}})^\beta(u+c\alpha\lambda^{\frac{1}{\beta}})(-s+\beta)_2F_1[1,1-s+\beta,2-s+\beta,\frac{-c\alpha\lambda^{\frac{1}{\beta}}}{u}]}{1-s+\beta}\\
&\left.+\frac{(\alpha\lambda^{\frac{1}{\beta}})^\beta u(-1+s-\beta)}{1-s+\beta}\right)\\
k_8^*(\lambda,\alpha,\beta,u,v,-s) :=\ & \frac{\alpha\lambda}{c^2(1+s)}\left(\frac{(\frac{1}{1-\alpha})^s(c(-1-s+\alpha+s\alpha)-s(u+c-c\alpha)_2F_1[1,1+s,2+s,\frac{u}{c(-1+\alpha)}])}{(-1+\alpha)(u+c-c\alpha)}\right.\\
&\left.-\frac{(\alpha\lambda^{\frac{-1}{\beta}})^{1+s}(c(1+s)\alpha\lambda^{\frac{1}{\beta}}-s(u+c\alpha\lambda^{\frac{1}{\beta}})_2F_1[1,1+s,2+s,\frac{-u\alpha\lambda^{\frac{-1}{\beta}}}{c}])])}{u+c\alpha\lambda^{\frac{1}{\beta}}}\right)\\
&+\frac{1}{c^2(1+s-\beta)}\left(-\frac{(\frac{1}{1-\alpha})^{s-\beta}(c(1+s-\beta))}{(u+c-c\alpha)}+\frac{(\frac{1}{1-\alpha})^{s-\beta}(s-\beta)}{(-1+\alpha)}\right.\end{aligned}$$

$$ {}_2F_1[1,1+s-\beta,2+s-\beta,\frac{u}{c(-1+\alpha)}]+\frac{(\alpha\lambda^{\frac{-1}{\beta}})^{1+s-\beta}(-c\alpha\lambda^{\frac{1}{\beta}}(-1+s+\beta))}{u+c\alpha\lambda^{\frac{1}{\beta}}} +(\alpha\lambda^{\frac{-1}{\beta}})^{1+s-\beta}(-s+\beta){}_2F_1[1,1+s-\beta,2+s-\beta,-\frac{u\alpha\lambda^{\frac{-1}{\beta}}}{c}]\Bigg) $$

$$ k_9^*(\lambda,\alpha,\beta,u,v,-s):=\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}}\frac{-(t^\beta-\alpha\lambda)}{(u(1-t)+(u+c)t)^2}(1-t)^{-s}dt $$

$$ k_{10}^*(\lambda,\alpha,\beta,u,v,-s):=\int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha}\frac{t^\beta-\alpha\lambda}{(u(1-t)+(u+c)t)^2}(1-t)^{-s}dt $$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}}\geq 1-\alpha$, then

$$ \begin{aligned} &\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt \\ \leq\ &\int_0^{1-\alpha}\frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}[t^{-s}|f'(u)|^\mu+(1-t)^{-s}|f'(v)|^\mu]dt \\ =\ &\int_0^{1-\alpha}\frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}t^{-s}dt|f'(u)|^\mu+\int_0^{1-\alpha}\frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}(1-t)^{-s}dt|f'(v)|^\mu \\ =\ &k_{11}^*(\lambda,\alpha,\beta,u,v,-s)|f'(u)|^\mu+k_{12}^*(\lambda,\alpha,\beta,u,v,-s)|f'(v)|^\mu. \end{aligned} \tag{3.53} $$

where

$$ k_{11}^*(\lambda,\alpha,\beta,u,v,-s):=\frac{(1-\alpha)^{-s}}{u^2(u-c\alpha+c)}\Bigg((1-\alpha)\alpha\lambda u-(1-\alpha)^{1+\beta}u +\frac{(-1+\alpha)\alpha\lambda s(u+c-c\alpha){}_2F_1[1,1-s,2-s,\frac{c(-1+\alpha)}{u}])}{s-1} +\frac{(1-\alpha)^{1+\beta}(u+c-c\alpha)(s-\beta){}_2F_1[1,1-s+\beta,2-s+\beta,\frac{c(-1+\alpha)}{u}]}{-1+s-\beta}\Bigg) $$

$$ k_{12}^*(\lambda,\alpha,\beta,u,v,-s):=\int_0^{1-\alpha}\frac{-(t^\beta-\alpha\lambda)}{(u(1-t)+(u+c)t)^2}(1-t)^{-s}dt $$

(d) (i) If $1+(\lambda(\alpha-1))^{\frac{1}{\beta}}\leq 1-\alpha$, then

$$ \begin{aligned} &\int_{1-\alpha}^1\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt \\ \leq\ &\int_{1-\alpha}^1\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}[t^{-s}|f'(u)|^\mu+(1-t)^{-s}|f'(v)|^\mu]dt \\ =\ &\int_{1-\alpha}^1\frac{(t-1)^\beta-\lambda(\alpha-1)}{(\bar{A}_t)^2}t^{-s}dt|f'(u)|^\mu \\ &+\int_{1-\alpha}^1\frac{(t-1)^\beta-\lambda(\alpha-1)}{(\bar{A}_t)^2}(1-t)^{-s}dt|f'(v)|^\mu \\ =\ &k_{13}^*(\lambda,\alpha,\beta,u,v,-s)|f'(u)|^\mu+k_{14}^*(\lambda,\alpha,\beta,u,v,-s)|f'(v)|^\mu. \end{aligned} \tag{3.54} $$

where

$$ k_{13}^*(\lambda,\alpha,\beta,u,v,-s):=\frac{(-1)^\beta(1-\alpha)^{-s}F_1[1-s,-\beta,2,2-s,1-\alpha,\frac{c(-1+\alpha)}{u}]}{u^2(-1+s)} +\frac{(-1)^\beta(1-\alpha)^{-s}\alpha F_1[1-s,-\beta,2,2-s,1-\alpha,\frac{c(-1+\alpha)}{u}]}{u^2(-1+s)} +\frac{1}{c^2(1+s)}(1-\alpha)\lambda\Bigg(\frac{-c(1+s)+(u+c)s{}_2F_1[1,1+s,2+s,-\frac{u}{c}]}{u+c} $$

$$+\frac{(1-\alpha)^{-s-1}(-c(s+1)(\alpha-1)-s(u+c-c\alpha)_2F_1[1,1+s,2+s,\frac{u}{c(-1+\alpha)}])}{u+c-c\alpha}\Bigg)$$

$$+\frac{(-1)^{\beta}\Gamma(1-s)\Gamma(1+\beta)_2F_1[2,1-s,2-s+\beta,-\frac{c}{u}]}{u^2\Gamma(2-s+\beta)}$$

$$k_{14}^*(\lambda,\alpha,\beta,u,v,-s):=\int_{1-\alpha}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{((1-t)v+t(u+c))^2}(1-t)^{-s}dt$$

(*ii*) If $1+(\lambda(\alpha-1))^{\frac{1}{\beta}}\geq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\\
\leq\ &\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2}[t^{-s}|f'(u)|^{\mu}+(1-t)^{-s}|f'(v)|^{\mu}]dt\\
&+\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}[t^{-s}|f'(u)|^{\mu}+(1-t)^{-s}|f'(v)|^{\mu}]dt\\
=\ &\Bigg[\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2}t^{-s}dt\\
&+\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}t^{-s}dt\Bigg]|f'(u)|^{\mu}\\
&+\Bigg[\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2}(1-t)^{-s}dt\\
&+\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}(1-t)^{-s}dt\Bigg]|f'(v)|^{\mu}\\
=\ &[k_{15}^*(\lambda,\alpha,\beta,u,v,-s)+k_{16}^*(\lambda,\alpha,\beta,u,v,-s)]|f'(u)|^{\mu}\\
&+[k_{17}^*(\lambda,\alpha,\beta,u,v-s)+k_{18}^*(\lambda,\alpha,\beta,u,v,-s)]|f'(v)|^{\mu}.
\end{aligned}\tag{3.55}$$

where

$$\begin{aligned}
k_{15}^*(\lambda,\alpha,\beta,u,v,-s):=&\frac{-1}{u^2(-1+s)}\alpha^{-\beta}(-((-1+\alpha)\lambda)^{\frac{1}{\beta}})^{-\beta}((-(-1+\alpha)\\
&\times(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}}))^{-s})\Bigg(-(-1+\alpha)(\alpha((-1+\alpha)\lambda)^{\frac{1}{\beta}})^{\beta}\\
&\times F_1[1-s,-\beta,2,2-s,1-\alpha,\frac{c}{-1+\alpha}u]\\
&-(1-\alpha)^{s}\alpha^{\beta}(((-1+\alpha)\lambda)^{\frac{1}{\beta}})^{\beta}(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})\\
&\times F_1[1-s,-\beta,2,2-s,1+((-1+\alpha)\lambda)^{\frac{1}{\beta}},\frac{-c(1+((-1+\alpha)\lambda))^{\frac{1}{\beta}}}{u}]\Bigg)+\frac{\lambda(\alpha-1)}{c^2(1+s)}\\
&\times\Bigg((1-\alpha)^{-1-s}\Bigg(\frac{-c(1+s)(-1+\alpha)}{u+c-c\alpha}-s_2F_1[1,1+s,2+s,\frac{u}{(c(-1+\alpha))}]\Bigg)\\
&+\left(\frac{1}{(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})}\right)^{1+s}\Bigg(\frac{-c(1+s)(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})}{u+c+c((-1+\alpha)\lambda)^{\frac{1}{\beta}}}\\
&+s_2F_1[1,1+s,2+s,-\frac{u}{c(1+((-1+\alpha)\lambda)^{\frac{1}{\beta}})}]\Bigg)\Bigg)
\end{aligned}$$

$$k_{16}^*(\lambda,\alpha,\beta,u,v,-s):=\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{((1-t)u+t(u+c))^2}t^{-s}dt$$

$$k_{17}^*(\lambda,\alpha,\beta,u,v,-s):=\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{((1-t)u+t(u+c))^2}(1-t)^{-s}dt$$

$$k^*_{18}(\lambda,\alpha,\beta,u,v,-s):=\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{((1-t)u+t(u+c))^2}(1-t)^{-s}dt$$

Where $c=\eta(v,u)$. By substituting (3.9) to (3.12) and (3.52) to (3.55) in equation (3.8) gives the required result. □

Corollary 10 *Assuming that* $f:M=[u,u+\eta(v,u)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f'\in L_1[u,u+\eta(v,u)]$ *for* $u,u+\eta(v,u)\in M$ *with* $u<u+\eta(v,u)$*. If* $|f'|^{\mu}$ *be s-harmonic Godunova-Levin preinvex on* M *for* $\mu>1$ *with* $\frac{1}{\gamma}+\frac{1}{\mu}=1$*, then*

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{u}^{u+\eta(v,u)}\frac{f(z)}{z^2}dz-f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}[s_5^{\frac{1}{\gamma}}(\lambda,u,v)\{s^*_1(\lambda,u,v,-s)|f'(u)|^{\mu}$$
$$+s^*_2(\lambda,u,v,-s)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}+s_6^{\frac{1}{\gamma}}(\lambda,u,v)\{s^*_3(\lambda,u,v,-s)$$
$$|f'(u)|^{\mu}+s^*_4(\lambda,u,v,-s)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].$$

where

$$s^*_1(\lambda,u,v,-s):=\frac{2^{-2+s}\left(\frac{2u}{2u+c}+(-1+s)\Gamma(2-s)_2\tilde{F}_1[1,2-s,3-s,-\frac{c}{2u}]\right)}{u^2}$$
$$s^*_2(\lambda,u,v,-s):=\frac{F_1[2,s,2,3,\frac{1}{2},-\frac{c}{2u}]}{8u^2}$$
$$s^*_3(\lambda,u,v,-s):=\frac{u(-1+s)-(u-us-cs)_2F_1[1,1-s,2-s,-\frac{c}{u}]}{u^2c(-1+s)}+\frac{2^{-1+s}(-2(u+c))}{uc(2u+c)}+\frac{(-u+us+cs)_2F_1[1,1-s,2-s,-\frac{c}{2u}]}{u^2c(-1+s)}$$
$$s^*_4(\lambda,u,v,-s):=\int_{\frac{1}{2}}^{1}\frac{-(t-1)(1-t)^{-s}}{((1-t)u+t(u+c))^2}dt;\ where\ \ c=\eta(v,u)$$

Also, $s_5(\lambda,u,v)$ *and* $s_6(\lambda,u,v)$ *are defined in Corollary* 4.

Proof. From (3.7), we have

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{u}^{u+\eta(v,u)}\frac{f(z)}{z^2}dz-f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\frac{|t-1|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right].$$

By using power mean integral inequality,

$$\leq u\eta(v,u)\{u+\eta(v,u)\}$$
$$\left[\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\frac{|t-1|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^{1}\frac{|t-1|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right]$$

Since $|f'|^\mu$ be s-harmonic Godunova-Levin preinvex function on the interval $[u, u+\eta(v,u)]$ for $\mu > 1$ and $s \in (0,1]$, as $t \in [0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^\mu \le t^{-s}|f'(u)|^\mu + (1-t)^{-s}|f'(v)|^\mu$$

$$\begin{aligned}
&\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right| \\
\le\ & u\eta(v,u)\{u+\eta(v,u)\}\Bigg[\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}[t^{-s}|f'(u)|^\mu+(1-t)^{-s}|f'(v)|^\mu]dt\right)^{\frac{1}{\mu}} \\
&+\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(\bar{A}_t)^2}[t^{-s}|f'(u)|^\mu+(1-t)^{-s}|f'(v)|^\mu]dt\right)^{\frac{1}{\mu}}\Bigg] \\
=\ & u\eta(v,u)\{u+\eta(v,u)\}\Bigg[\left(\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{t^{-s+1}}{(\bar{A}_t)^2}dt|f'(u)|^\mu+\int_0^{\frac{1}{2}}\frac{(t)(1-t)^{-s}}{(\bar{A}_t)^2}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}} \\
&+\left(\int_{\frac{1}{2}}^1\frac{-(t-1)}{(\bar{A}_t)^2}dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^1\frac{-(t-1)t^{-s}}{(\bar{A}_t)^2}dt|f'(u)|^\mu+\int_{\frac{1}{2}}^1\frac{-(t-1)(1-t)^{-s}}{(\bar{A}_t)^2}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}\Bigg] \\
\le\ & u\eta(v,u)\{u+\eta(v,u)\}[s_5^{\frac{1}{\gamma}}(\lambda,u,v)\{s_1^*(\lambda,u,v,-s)|f'(u)|^\mu+s_2^*(\lambda,u,v,-s)|f'(v)|^\mu\}^{\frac{1}{\mu}} \\
&+s_6^{\frac{1}{\gamma}}(\lambda,u,v)\{s_3^*(\lambda,u,v,-s)|f'(u)|^\mu+s_4^*(\lambda,a,b,-s)|f'(v)|^\mu\}^{\frac{1}{\mu}}].
\end{aligned}$$

□

Theorem 11 *Assuming that $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[u, u+\eta(v,u)]$ for $u, u+\eta(v,u) \in M$ with $u < u+\eta(v,u)$, $\beta \in (0,1]$ and $\lambda, \alpha \in [0,1]$. If $|f'|^\mu$ is s-harmonic Godunova-Levin preinvex on M for $\mu > 1$, we have*

(a) *If $(\alpha\lambda)^{\frac{1}{\beta}} \le 1-\alpha \le 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, then*

$$\begin{aligned}
|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \le\ & u\eta(v,u)\{u+\eta(v,u)\}\Bigg[(k_{19}(\lambda,\alpha,\beta,u,v,\gamma)+k_{20}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}} \\
&\left((1-\alpha)\frac{\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu+|f'(v)|^\mu\right\}}{1-s}\right)^{\frac{1}{\mu}}+(k_{23}(\lambda,\alpha,\beta,u,v,\gamma) \\
&+k_{24}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}\left(\alpha\frac{\left\{|f'(u)|^\mu+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu\right\}}{1-s}\right)^{\frac{1}{\mu}}\Bigg].
\end{aligned}$$

(b) *If $(\alpha\lambda)^{\frac{1}{\beta}} \le 1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \le 1-\alpha$, then*

$$\begin{aligned}
|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \le\ & u\eta(v,u)\{u+\eta(v,u)\}\Bigg[(k_{19}(\lambda,\alpha,\beta,u,v,\gamma)+k_{20}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}} \\
&\left((1-\alpha)\frac{\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu+|f'(v)|^\mu\right\}}{1-s}\right)^{\frac{1}{\mu}}
\end{aligned}$$

$$+(k_{22}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}\left(\alpha\frac{\left\{|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}}{1-s}\right)^{\frac{1}{\mu}}\Bigg].$$

(c) *If* $1-\alpha \leq (\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$*, then*

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}\Bigg[(k_{21}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}$$

$$\left((1-\alpha)\frac{\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(v)|^{\mu}\right\}}{1-s}\right)^{\frac{1}{\mu}}+(k_{23}(\lambda,\alpha,\beta,u,v,\gamma)$$

$$+k_{24}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}\left(\alpha\frac{\left\{|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}}{1-s}\right)^{\frac{1}{\mu}}\Bigg].$$

where $k_{19}(\lambda,\alpha,\beta,u,v,\gamma)$ *to* $k_{24}(\lambda,\alpha,\beta,u,v,\gamma)$ *are defined in Theorem* 5.

Proof. By using Lemma 2 and Hölder's integral inequality, we have

Similarly to the process of (3.17) to (3.21) and we obtain $k_{19},\ k_{20},\ k_{21},\ k_{22},\ k_{23}\ and\ k_{24}$.

(c) Consider,

$$\int_0^{1-\alpha}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt$$

Now similarly to the process of (3.22) to (3.23).

Using Hermite-Hadamard's inequality for s-harmonic Godunova-Levin preinvex functions,

$$\begin{aligned}
&\int_0^{1-\alpha}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\\
\leq\ & \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\left(\frac{\{u+\eta(v,u)\}-\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}{\{u+\eta(v,u)\}\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\right)\\
&\left[\frac{|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}+|f'(u+\eta(v,u))|^{\mu}}{1-s}\right]\\
=\ & \frac{\bar{A}_{1-\alpha}-u}{\eta(v,u)}\left[\frac{|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}+|f'(u+\eta(v,u))|^{\mu}}{1-s}\right]\\
=\ & \frac{\alpha u+(1-\alpha)(u+\eta(v,u))-u}{\eta(v,u)}\left[\frac{|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}+|f'(u+\eta(v,u))|^{\mu}}{1-s}\right]\\
\leq\ & (1-\alpha)\left[\frac{|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}+|f'(v)|^{\mu}}{1-s}\right] \qquad (3.56)
\end{aligned}$$

Above Inequality holds for $\alpha = 1$.

(d) Consider,

$$\int_{1-\alpha}^{1}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt$$

Similarly to the process of (3.25) to (3.26).

Using Hermite-Hadamard's inequality for s-harmonic Godunova Levin preinvex functions, then

$$\begin{aligned}
&\int_{1-\alpha}^{1}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\\
\leq\ & \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\left(\frac{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}-u}{\frac{u^2\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\right)\left[\frac{|f'(u)|^{\mu}+|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}}{1-s}\right]\\
=\ & \frac{\{u+\eta(v,u)\}-\bar{A}_{1-\alpha}}{\eta(v,u)}\left[\frac{|f'(u)|^{\mu}+|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}}{1-s}\right]\\
=\ & \frac{\{u+\eta(v,u)\}-\alpha u-(1-\alpha)(u+\eta(v,u))}{\eta(v,u)}\left[\frac{|f'(u)|^{\mu}+|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}}{1-s}\right]\\
\leq\ & \alpha\left[\frac{|f'(u)|^{\mu}+|f'(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}})|^{\mu}}{1-s}\right] \qquad (3.57)
\end{aligned}$$

Above Inequality holds for $\alpha = 0$.

By substituting (3.18) to (3.21), (3.56) and (3.57) in equation (3.17) gives the required result. □

Corollary 11 *Assuming that $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[u, u+\eta(v,u)]$ for $u, u+\eta(v,u) \in M$ with $u < u+\eta(v,u)$. If $|f'|^{\mu}$ is s-harmonic Godunova-Levin preinvex on M for $\mu > 1$, then*

$$\begin{aligned}
&\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{u}^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|\\
&\leq u\eta(v,u)\{u+\eta(v,u)\}\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma}}\\
&[s_7^*(\lambda,u,v,-s,\mu)|f'(u)|^{\mu}+s_8^*(\lambda,u,v,-s,\mu)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}\\
&+\{s_9^*(\lambda,u,v,-s,\mu)|f'(u)|^{\mu}+s_{10}^*(\lambda,u,v,-s,\mu)|f'(v)|^{\mu}\}^{\frac{1}{\mu}}].
\end{aligned}$$

where

$$\begin{aligned}
s_7^*(\lambda,u,v,-s,\mu) &:= u^{-2\mu}(-\frac{c}{u})^{-1+s}B[-\frac{c}{2u},1-s,1-2\mu]\\
s_8^*(\lambda,u,v,-s,\mu) &:= -\frac{u^{1-2\mu}{}_2F_1[1,2-2\mu-s,2-2\mu,\frac{u}{u+c}]}{(u+c)(1-2\mu)}\\
&\quad+\frac{2^{-2+2\mu+s}(2u+c)^{1-2\mu}{}_2F_1[1,2-2\mu-s,2-2\mu,\frac{2u+c}{2(u+c)}]}{(u+c)(1-2\mu)}\\
s_9^*(\lambda,u,b,-s,\mu) &:= \frac{(u+c)^{1-2\mu}{}_2F_1[1,2-2\mu-s,2-s,-\frac{c}{u}]}{u-us}\\
&\quad-\frac{2^{-2+2\mu-s}(2u+c)^{1-2\mu}{}_2F_1[1,2-2\mu-s,2-s,-\frac{c}{2u}]}{u-us}\\
s_{10}^*(\lambda,u,b,-s,\mu) &:= \frac{(u+\frac{c}{2})^{-2\mu}(\frac{c}{u+c})^{s}(u+c)^{-2\mu}}{2c(-1+2\mu)\Gamma(2-2\mu-s)}(-2(u+\frac{c}{2})^{2\mu}\\
&\quad(u+c)\Gamma(2-2\mu)\Gamma(1-s)+(u+c)^{2\mu}(2u+c)
\end{aligned}$$

$$\Gamma(2-2\mu-s)_2F_1[1-2\mu,s,2-2\mu,\frac{2u+c}{2(u+c)}]); \ where \ \ c=\eta(v,u).$$

Proof. From (3.7), we have

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt + \int_{\frac{1}{2}}^1\frac{|(t-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right]$$

Since $|f'|^\mu$ be s-harmonic Godunova-Levin preinvex function on the interval $[u,u+\eta(v,u)]$ for $\mu>1$ and $s\in(0,1]$, as $t\in[0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^\mu \leq t^{-s}|f'(u)|^\mu+(1-t)^{-s}|f'(v)|^\mu$$

Using Hölder's integral inequality, we have

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left(\int_0^{\frac{1}{2}}t^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{1}{(\bar{A}_t)^{2\mu}}(t^{-s}|f'(u)|^\mu+(1-t)^{-s}|f'(v)|^\mu)dt\right)^{\frac{1}{\mu}}$$
$$+\left(\int_{\frac{1}{2}}^1(1-t)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^1\frac{1}{(\bar{A}_t)^{2\mu}}(t^{-s}|f'(u)|^\mu+(1-t)^{-s}|f'(v)|^\mu)dt\right)^{\frac{1}{\mu}}$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left(\int_0^{\frac{1}{2}}t^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{t^{-s}}{(\bar{A}_t)^{2\mu}}dt|f'(u)|^\mu+\int_0^{\frac{1}{2}}\frac{(1-t)^{-s}}{(\bar{A}_t)^{2\mu}}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}$$
$$+\left(\int_{\frac{1}{2}}^1(1-t)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^1\frac{t^{-s}}{(\bar{A}_t)^{2\mu}}dt|f'(u)|^\mu+\int_{\frac{1}{2}}^1\frac{(1-t)^{-s}}{(\bar{A}_t)^{2\mu}}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma}}[\{s_7^*(\lambda,u,v,-s,\mu)|f'(u)|^\mu+s_8^*(\lambda,u,v,-s,\mu)|f'(v)|^\mu\}^{\frac{1}{\mu}}$$
$$+\{s_9^*(\lambda,u,v,-s,\mu)|f'(u)|^\mu+s_{10}^*(\lambda,u,v,-s,\mu)|f'(v)|^\mu\}^{\frac{1}{\mu}}].$$

□

Theorem 12 *Assuming that* $f: M=[u,u+\eta(v,u)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ *is differentiable on the interior ,* M^o *of* M *where* $f'\in L_1[u,u+\eta(v,u)]$ *for* $u, u+\eta(v,u)\in M$ *with* $u<u+\eta(v,u), \beta\in(0,1]$ *and* $\lambda,\alpha\in[0,1]$. *If* $|f'|$ *is s-harmonic Godunova-Levin preinvex on* M, *we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}}\leq 1-\alpha\leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}$$
$$[\{\{k_7^*(\lambda,\alpha,\beta,u,v,-s)+k_8^*(\lambda,\alpha,\beta,u,v,-s)\}$$
$$+\{k_{15}^*(\lambda,\alpha,\beta,u,v,-s)+k_{16}^*(\lambda,\alpha,\beta,u,v,-s)\}\}|f'(u)|$$
$$+\{\{k_9^*(\lambda,\alpha,\beta,u,v,-s)+k_{10}^*(\lambda,\alpha,\beta,u,v,-s)\}$$
$$+\{k_{17}^*(\lambda,\alpha,\beta,u,v,-s)+k_{18}^*(\lambda,\alpha,\beta,u,v,-s)\}\}|f'(v)|].$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1 + (\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\beta,\alpha,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\} \\ & [\{\{k_7^*(\lambda,\alpha,\beta,u,v,-s)+k_8^*(\lambda,\alpha,\beta,u,v,-s)\} \\ & +k_{13}^*(\lambda,\alpha,\beta,u,v,-s)\}|f'(u)|+\{\{k_9^*(\lambda,\alpha,\beta,u,v,-s) \\ & +k_{10}^*(\lambda,\alpha,\beta,u,v,-s)\}+k_{14}^*(\lambda,\alpha,\beta,u,v,-s)\}|f'(v)|].\end{aligned}$$

(c) *If* $1-\alpha \leq (\alpha\lambda)^{\frac{1}{\beta}} \leq 1 + (\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$\begin{aligned}|\Psi_f(\lambda,\beta,\alpha,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\} \\ & [\{\{k_{15}^*(\lambda,\alpha,\beta,u,v,-s)+k_{16}^*(\lambda,\alpha,\beta,u,v,-s)\} \\ & +k_{11}^*(\lambda,\alpha,\beta,u,v,-s)\}|f'(u)|+\{\{k_{17}^*(\lambda,\alpha,\beta,u,v,-s) \\ & +k_{18}^*(\lambda,\alpha,\beta,u,v,-s)\}+k_{12}^*(\lambda,\alpha,\beta,u,v,-s)\}|f'(v)|].\end{aligned}$$

where $k_7^*(\lambda,\alpha,\beta,u,v,-s)$ *to* $k_{18}^*(\lambda,\alpha,\beta,u,v,-s)$ *are defined in Theorem* 10.

Proof. By using Lemma 2, we have

$$\begin{aligned}|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right. \\ & \left.+\int_{1-\alpha}^1\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right]\end{aligned}$$

Since $|f'|$ be s-harmonic Godunova-Levin preinvex function on the interval $[u, u+\eta(v,u)]$ and $s \in (0,1]$, as $t \in [0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right| \leq t^{-s}|f'(u)| + (1-t)^{-s}|f'(v)|$$

$$\begin{aligned}& |\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \\ \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}[t^{-s}|f'(u)|+(1-t)^{-s}|f'(v)|]dt\right. \\ & \left.+\int_{1-\alpha}^1\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}[t^{-s}|f'(u)|+(1-t)^{-s}|f'(v)|]dt\right] \\ \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\left\{\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}t^{-s}dt|f'(u)|\right.\right. \\ & \left.+\int_{1-\alpha}^1\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}t^{-s}dt|f'(v)|\right\} \\ & +\left\{\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}(1-t)^{-s}dt|f'(u)|\right. \\ & \left.\left.+\int_{1-\alpha}^1\frac{|(t-1)^\beta-\lambda(\alpha-1)|}{(\bar{A}_t)^2}(1-t)^{-s}dt|f'(v)|\right\}\right] \qquad (3.58)\end{aligned}$$

(a) (i) If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\int_0^{1-\alpha}\frac{|t^\beta-\alpha\lambda|}{(\bar{A}_t)^2}t^{-s}dt$$

$$= \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} t^{-s} dt + \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta - \alpha\lambda}{(\bar{A}_t)^2} t^{-s} dt$$
$$= k_7^*(\lambda, \alpha, \beta, u, v, -s) + k_8^*(\lambda, \alpha, \beta, u, v, -s). \tag{3.59}$$

and

$$\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} (1-t)^{-s} dt$$
$$= \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} (1-t)^{-s} dt + \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta - \alpha\lambda}{(\bar{A}_t)^2} (1-t)^{-s} dt$$
$$= k_9^*(\lambda, \alpha, \beta, u, v, -s) + k_{10}^*(\lambda, \alpha, \beta, u, v, -s). \tag{3.60}$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}} \geq 1 - \alpha$, then

$$\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} t^{-s} dt$$
$$= \int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} t^{-s} dt$$
$$= k_{11}^*(\lambda, \alpha, \beta, u, v, -s) \tag{3.61}$$

and

$$\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} (1-t)^{-s} dt$$
$$= \int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} (1-t)^{-s} dt$$
$$= k_{12}^*(\lambda, \alpha, \beta, u, v, -s). \tag{3.62}$$

(b) (i) If $1 + (\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1 - \alpha$, then

$$\int_{1-\alpha}^1 \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2} t^{-s} dt$$
$$= \int_{1-\alpha}^1 \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2} t^{-s} dt$$
$$= k_{13}^*(\lambda, \alpha, \beta, u, v, -s) \tag{3.63}$$

and

$$\int_{1-\alpha}^1 \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2} (1-t)^{-s} dt$$
$$= \int_{1-\alpha}^1 \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2} (1-t)^{-s} dt$$
$$= k_{14}^*(\lambda, \alpha, \beta, u, v, -s). \tag{3.64}$$

(ii) If $1 + (\lambda(\alpha-1))^{\frac{1}{\beta}} \geq 1 - \alpha$, then

$$\int_{1-\alpha}^1 \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2} t^{-s} dt$$
$$= \int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^\beta - \lambda(\alpha-1))}{(\bar{A}_t)^2} t^{-s} dt$$
$$+ \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^1 \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2} t^{-s} dt$$

$$= k^*_{15}(\lambda,\alpha,\beta,u,v,-s)+k^*_{16}(\lambda,\alpha,\beta,u,v,-s) \tag{3.65}$$

and

$$\begin{aligned}
&\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}(1-t)^{-s}dt\\
=&\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2}(1-t)^{-s}dt\\
&+\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}(1-t)^{-s}dt\\
=&\ k^*_{17}(\lambda,\alpha,\beta,u,v,-s)+k^*_{18}(\lambda,\alpha,\beta,u,v,-s). \qquad (3.66)
\end{aligned}$$

By substituting (3.59) to (3.66) in equation (3.58) gives the required result. □

Harmonic P-preinvex Functions

Theorem 13 *Assuming that $f: M=[u,u+\eta(v,u)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ is differentiable on the interior, M^o of M where $f'\in L_1[u,u+\eta(v,u)]$ for $u,u+\eta(v,u)\in M$ with $u<u+\eta(v,u)$, $\beta\in(0,1]$ and $\lambda,\alpha\in[0,1]$. If $|f'|^{\mu}$ is harmonic P-preinvex on M for $\mu>1$ with $\frac{1}{\gamma}+\frac{1}{\mu}=1$, we have*

(a) *If $(\alpha\lambda)^{\frac{1}{\beta}}\leq 1-\alpha\leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, then*

$$\begin{aligned}
|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))|\leq\ & u\eta(v,u)\{u+\eta(v,u)\}[(k_1(\lambda,\alpha,\beta,u,v)+k_2(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\\
&\times\{(k^{**}_7(\lambda,\alpha,\beta,u,v,0)+k^{**}_8(\lambda,\alpha,\beta,u,v,0))[|f'(u)|^{\mu}+|f'(v)|^{\mu}]\}^{\frac{1}{\mu}}\\
&+(k_5(\lambda,\alpha,\beta,u,v)+k_6(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(k^{**}_{11}(\lambda,\alpha,\beta,u,v,0)\\
&+k^{**}_{12}(\lambda,\alpha,\beta,u,v,0))[|f'(u)|^{\mu}+|f'(v)|^{\mu}]\}^{\frac{1}{\mu}}].
\end{aligned}$$

(b) *If $(\alpha\lambda)^{\frac{1}{\beta}}\leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}\leq 1-\alpha$, then*

$$\begin{aligned}
|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))|\leq\ & u\eta(v,u)\{u+\eta(v,u)\}[(k_1(\lambda,\alpha,\beta,u,v)+k_2(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\\
&\times\{(k^{**}_7(\lambda,\alpha,\beta,u,v,0)+k^{**}_8(\lambda,\alpha,\beta,u,v,0))\}[|f'(u)|^{\mu}+|f'(v)|^{\mu}]\}^{\frac{1}{\mu}}\\
&+(k_4(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{(k^{**}_{10}(\lambda,\alpha,\beta,u,v,0)[|f'(u)|^{\mu}+|f'(v)|^{\mu}]\}^{\frac{1}{\mu}}].
\end{aligned}$$

(c) *If $1-\alpha\leq(\alpha\lambda)^{\frac{1}{\beta}}\leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, then*

$$\begin{aligned}
|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))|\leq\ & u\eta(v,u)\{u+\eta(v,u)\}[(k_3(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\{k^{**}_9(\lambda,\alpha,\beta,u,v,0)\\
&\times[|f'(u)|^{\mu}+|f'(v)|^{\mu}]\}^{\frac{1}{\mu}}+(k_5(\lambda,\alpha,\beta,u,v)+k_6(\lambda,\alpha,\beta,u,v))^{\frac{1}{\gamma}}\\
&\times\{(k^{**}_{11}(\lambda,\alpha,\beta,u,v,0)+k^{**}_{12}(\lambda,\alpha,\beta,u,v,0))[|f'(u)|^{\mu}+|f'(v)|^{\mu}]\}^{\frac{1}{\mu}}].
\end{aligned}$$

Proof. By using Lemma 2 and power mean integral inequality,
Similarly to the process of (3.8) to (3.12) and we obtain $k_1,\ k_2,\ k_3,\ k_4,\ k_5$ *and* k_6. Since $|f'|^{\mu}$ be harmonic P-preinvex function on the interval $[u,u+\eta(v,u)]$ for $\mu\in(1,\infty)$, as $t\in[0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^{\mu}\leq|f'(u)|^{\mu}+|f'(v)|^{\mu}$$

(c) (i) If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\begin{aligned}
&\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt \\
\leq\ & \int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}[|f'(u)|^\mu + |f'(v)|^\mu]dt \\
&+ \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta-\alpha\lambda}{(\bar{A}_t)^2}[|f'(u)|^\mu + |f'(v)|^\mu]dt \\
=\ & \left[\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}dt + \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta-\alpha\lambda}{(\bar{A}_t)^2}dt\right][|f'(u)|^\mu + |f'(v)|^\mu] \\
=\ & \{k_7^{**}(\lambda,\alpha,\beta,u,v,0) + k_8^{**}(\lambda,\alpha,\beta,u,v,0)\}[|f'(u)|^\mu + |f'(v)|^\mu]. \qquad (3.67)
\end{aligned}$$

where

$$\begin{aligned}
k_7^{**}(\lambda,\alpha,\beta,u,v,0) :=& \frac{-\alpha\lambda^{\frac{1}{\beta}}(1+\beta)((\alpha\lambda^{\frac{1}{\beta}})^\beta - \alpha\lambda)}{u(u+c\alpha\lambda^{\frac{1}{\beta}}(1+\beta))} \\
&+ \frac{(\alpha\lambda^{\frac{1}{\beta}})^{\beta+1}(u+c\alpha\lambda^{\frac{1}{\beta}})\beta_2F_1[1,1+\beta,2+\beta,-\frac{c\alpha\lambda^{\frac{1}{\beta}}}{u}]}{u^2(u+c\alpha\lambda^{\frac{1}{\beta}}(1+\beta))} \\
k_8^{**}(\lambda,\alpha,\beta,u,v,0) :=& \frac{(c(1-\alpha)^{\beta+1} + u\alpha\lambda)}{uc(u+c-c\alpha)} - \frac{(1-\alpha)^{\beta+1}\beta_2F_1[1,1+\beta,2+\beta,\frac{-c(1-\alpha)}{u}]}{u^2(1+\beta)} \\
&- \frac{(c(\alpha\lambda^{\frac{1}{\beta}})^{1+\beta} + u\alpha\lambda)}{uc(u+c\alpha\lambda^{\frac{1}{\beta}})} + \frac{(\alpha\lambda^{\frac{1}{\beta}})^{1+\beta}\beta_2F_1[1,1+\beta,2+\beta,-\frac{(c\alpha\lambda^{\frac{1}{\beta}})}{u}]}{u^2(1+\beta)}
\end{aligned}$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\begin{aligned}
&\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt \\
\leq\ & \int_0^{1-\alpha} \frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}[|f'(u)|^\mu + |f'(v)|^\mu]dt \\
\leq\ & \int_0^{1-\alpha} \frac{-(t^\beta-\alpha\lambda)}{(\bar{A}_t)^2}dt[|f'(u)|^\mu + |f'(v)|^\mu] \\
=\ & k_9^{**}(\lambda,\alpha,\beta,u,v,0)[|f'(u)|^\mu + |f'(v)|^\mu]. \qquad (3.68)
\end{aligned}$$

where

$$k_9^{**}(\lambda,\alpha,\beta,u,v,0) := \frac{(-1+\alpha)((1-\alpha)^\beta - \alpha\lambda)}{u(u+c-c\alpha)} - \frac{(1-\alpha)^\beta\beta_2F_1[1,1+\beta,2+\beta,\frac{c(-1+\alpha)}{u}]}{u^2(1+\beta)}$$

(d) (i) If $1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^1 \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^\mu dt \\
\leq\ & \int_{1-\alpha}^1 \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2}[|f'(u)|^\mu + |f'(v)|^\mu]dt \\
=\ & k_{10}^{**}(\lambda,\alpha,\beta,u,v,0)[|f'(u)|^\mu + |f'(v)|^\mu]. \qquad (3.69)
\end{aligned}$$

where

$$k_{10}^{**}(\lambda,\alpha,\beta,u,v,0) := \int_{1-\alpha}^1 \frac{(t-1)^\beta - \lambda(\alpha-1)}{((1-t)u+t(u+c))^2}dt$$

(*ii*) If $1+(\lambda(\alpha-1))^{\frac{1}{\beta}}\geq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\\
\leq\ &\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{(\bar{A}_t)^2}[|f'(u)|^{\mu}+|f'(v)|^{\mu}]dt\\
&+\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{(\bar{A}_t)^2}[|f'(u)|^{\mu}+|f'(v)|^{\mu}]dt\\
=\ &\{k_{11}^{**}(\lambda,\alpha,\beta,u,v,0)+k_{12}^{**}(\lambda,\alpha,\beta,u,v,0)\}[|f'(u)|^{\mu}+|f'(v)|^{\mu}]. \qquad (3.70)
\end{aligned}$$

where

$$\begin{aligned}
k_{11}^{**}(\lambda,\alpha,\beta,u,v,0)&:=\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}\frac{-((t-1)^{\beta}-\lambda(\alpha-1))}{((1-t)u+t(u+c))^2}dt\\
k_{12}^{**}(\lambda,\alpha,\beta,u,v,0)&:=\int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1}\frac{(t-1)^{\beta}-\lambda(\alpha-1)}{((1-t)u+t(u+c))^2}dt
\end{aligned}$$

Where $c=\eta(u,v)$. By substituting (3.9) to (3.12) and (3.67) to (3.70) in equation (3.8) gives the required result. □

Corollary 12 *Assuming that* $f: M=[u,u+\eta(v,u)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f'\in L_1[u,u+\eta(v,u)]$ *for* $u,u+\eta(v,u)\in M$ *with* $u<u+\eta(v,u)$ *.If* $|f'|^{\mu}$ *is harmonic P-preinvex on* M *for* $\mu>1$ *with* $\frac{1}{\gamma}+\frac{1}{\mu}=1$*, then*

$$\begin{aligned}
&\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{u}^{u+\eta(v,u)}\frac{f(z)}{z^2}dz-f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|\\
&\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\{s_5(\lambda,u,v)+s_6(\lambda,u,v)\}\left(|f'(u)|^{\mu}+|f'(v)|^{\mu}\right)^{\frac{1}{\mu}}\right].
\end{aligned}$$

where $s_5(\lambda,u,v)$ and $s_6(\lambda,u,v)$ are defined in Corollary 4.

Proof. From (3.7), we have

$$\begin{aligned}
&\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_{u}^{u+\eta(v,u)}\frac{f(z)}{z^2}dz-f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|\\
&\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right.\\
&\qquad\left.+\int_{\frac{1}{2}}^{1}\frac{|t-1|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right]
\end{aligned}$$

By power mean integral inequality, we have

$$\begin{aligned}
\leq\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}\frac{|t-1|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^{1}\frac{|t-1|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right]
\end{aligned}$$

Since $|f'|^{\mu}$ be harmonic P-preinvex function on the interval $[u,u+\eta(v,u)]$ for $\mu\in(1,\infty)$, as $t\in[0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^{\mu}\leq|f'(u)|^{\mu}+|f'(v)|^{\mu}$$

$$
\begin{aligned}
& \left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz - f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right| \\
\leq\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}[|f'(u)|^\mu+|f'(v)|^\mu]dt\right)^{\frac{1}{\mu}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(\bar{A}_t)^2}dt\right)^{1-\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^1\frac{|t-1|}{(\bar{A}_t)^2}[|f'(u)|^\mu+|f'(v)|^\mu]dt\right)^{\frac{1}{\mu}}\right] \\
=\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\left(\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}dt|f'(u)|^\mu+\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1\frac{-(t-1)}{(\bar{A}_t)^2}dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^1\frac{-(t-1)}{(\bar{A}_t)^2}dt|f'(u)|^\mu+\int_{\frac{1}{2}}^1\frac{-(t-1)}{(\bar{A}_t)^2}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}\right] \\
=\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\left(\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}dt\right)^{\frac{1}{\gamma}+\frac{1}{\mu}}(|f'(u)|^\mu+|f'(v)|^\mu)^{\frac{1}{\mu}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1\frac{-(t-1)}{(\bar{A}_t)^2}dt\right)^{\frac{1}{\gamma}+\frac{1}{\mu}}(|f'(u)|^\mu+|f'(v)|^\mu)^{\frac{1}{\mu}}\right] \\
=\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\left(\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}dt\right)(|f'(u)|^\mu+|f'(v)|^\mu)^{\frac{1}{\mu}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1\frac{-(t-1)}{(\bar{A}_t)^2}dt\right)(|f'(u)|^\mu+|f'(v)|^\mu)^{\frac{1}{\mu}}\right] \\
=\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\left(\int_0^{\frac{1}{2}}\frac{t}{(\bar{A}_t)^2}dt+\int_{\frac{1}{2}}^1\frac{-(t-1)}{(\bar{A}_t)^2}dt\right)(|f'(u)|^\mu+|f'(v)|^\mu)^{\frac{1}{\mu}}\right] \\
\leq\ & u\eta(v,u)\{u+\eta(v,u)\}\left[\{s_5(\lambda,u,v)+s_6(\lambda,u,v)\}(|f'(u)|^\mu+|f'(v)|^\mu)^{\frac{1}{\mu}}\right].
\end{aligned}
$$

□

Theorem 14 *Assuming that* $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ *is differentiable on the interior,* M^o *of* M *where* $f' \in L_1[u, u+\eta(v,u)]$ *for* $u, u+\eta(v,u) \in M$ *with* $u < u+\eta(v,u)$, $\beta \in (0,1]$ *and* $\lambda, \alpha \in [0,1]$. *If* $|f'|^\mu$ *is harmonic P-preinvex on* M *for* $\mu > 1$, *we have*

(a) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$
\begin{aligned}
|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\Bigg[(k_{19}(\lambda,\alpha,\beta,u,v,\gamma)+k_{20}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}} \\
& \times\left((1-\alpha)\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu+|f'(v)|^\mu\right\}\right)^{\frac{1}{\mu}}+(k_{23}(\lambda,\alpha,\beta,u,v,\gamma) \\
& +k_{24}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}\left(\alpha\left\{|f'(u)|^\mu+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu\right\}\right)^{\frac{1}{\mu}}\Bigg].
\end{aligned}
$$

(b) *If* $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, *then*

$$
\begin{aligned}
|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq\ & u\eta(v,u)\{u+\eta(v,u)\}\Bigg[(k_{19}(\lambda,\alpha,\beta,u,v,\gamma)+k_{20}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}} \\
& \times\left((1-\alpha)\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu+|f'(v)|^\mu\right\}\right)^{\frac{1}{\mu}} \\
& +(k_{22}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}\left(\alpha\left\{|f'(u)|^\mu+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu\right\}\right)^{\frac{1}{\mu}}\Bigg].
\end{aligned}
$$

(c) *If* $1-\alpha \leq (\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, *then*

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}\Bigg[(k_{21}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}} \times\left((1-\alpha)\left\{\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(v)|^{\mu}\right\}\right)^{\frac{1}{\mu}}+(k_{23}(\lambda,\alpha,\beta,u,v,\gamma) +k_{24}(\lambda,\alpha,\beta,u,v,\gamma))^{\frac{1}{\gamma}}\left(\alpha\left\{|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right\}\right)^{\frac{1}{\mu}}\Bigg].$$

where $k_{19}(\lambda,\alpha,\beta,u,v,\gamma)$ to $k_{24}(\lambda,\alpha,\beta,u,v,\gamma)$ are defined in Theorem 5.

Proof. By using Lemma 2 and Hölder's integral inequality, we have

Similarly to the process of (3.17) to (3.21) and we obtain $k_{19},\ k_{20},\ k_{21},\ k_{22},\ k_{23}$ *and* k_{24}.

(c) Consider,

$$\int_0^{1-\alpha}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt$$

Now similarly to the process of (3.22) to (3.23).

Using Hermite-Hadamard's inequality for harmonic P-preinvex functions, then we have

$$\begin{aligned}
&\int_0^{1-\alpha}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt \\
\leq\ & \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\left(\frac{\{u+\eta(v,u)\}-\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}{\{u+\eta(v,u)\}\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\right) \left[\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(u+\eta(v,u))|^{\mu}\right] \\
=\ & \frac{\bar{A}_{1-\alpha}-u}{\eta(v,u)}\left[\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(u+\eta(v,u))|^{\mu}\right] \\
=\ & \frac{\alpha u+(1-\alpha)(u+\eta(v,u))-u}{\eta(v,u)}\left[\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(u+\eta(v,u))|^{\mu}\right] \\
\leq\ & (1-\alpha)\left[\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}+|f'(v)|^{\mu}\right] \qquad (3.71)
\end{aligned}$$

Above Inequality holds for $\alpha=1$.

(d) Consider,

$$\int_{1-\alpha}^{1}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt$$

Similarly to the process of (3.25) to (3.26).

Using Hermite-Hadamard's inequality for harmonic P-preinvex functions, we have

$$\begin{aligned}
&\int_{1-\alpha}^{1}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|^{\mu}dt \\
\leq\ & \frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\left(\frac{\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}-u}{\frac{u^2\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}}\right)\left[|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right] \\
=\ & \frac{\{u+\eta(v,u)\}-\bar{A}_{1-\alpha}}{\eta(v,u)}\left[|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right] \\
=\ & \frac{\{u+\eta(v,u)\}-\alpha u-(1-\alpha)(u+\eta(v,u))}{\eta(v,u)}\left[|f'(u)|^{\mu}+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^{\mu}\right]
\end{aligned}$$

$$\leq \quad \alpha\left[|f'(u)|^\mu+\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_{1-\alpha}}\right)\right|^\mu\right] \tag{3.72}$$

Above Inequality holds for $\alpha=0$.

Where $c=\eta(u,v)$. By substituting (3.18) to (3.21), (3.71) and (3.72) in equation (3.17) gives the required result. ◻

Corollary 13 *Assuming that $f: M=[u,u+\eta(v,u)]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ is differentiable on the interior, M^o of M where $f'\in L_1[u,u+\eta(v,u)]$ for $u,u+\eta(v,u)\in M$ with $u<u+\eta(v,u)$. If $|f'|^\mu$ is harmonic P-preinvex on M for $\mu>1$, then*

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz-f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma}}$$
$$[\{s_1^{**}(\lambda,u,v,0,\mu)+s_2^{**}(\lambda,u,v,0,\mu)\}\left(|f'(u)|^\mu+|f'(v)|^\mu\right)]^{\frac{1}{\mu}}.$$

where

$$s_1^{**}(\lambda,u,v,0,\mu) \quad := \quad \frac{1}{u(2u+c)}$$
$$s_2^{**}(\lambda,u,v,0,\mu) \quad := \quad \frac{1}{(u+c)(2u+c)};\ \textit{where}\quad c=\eta(v,u).$$

Proof. From (3.7), we have

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz-f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{\frac{1}{2}}\frac{|t|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1\frac{|(t-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right]$$

Since $|f'|^\mu$ be harmonic P-preinvex function on the interval $[u,u+\eta(v,u)]$ for $\mu\in(1,\infty)$, as $t\in[0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right|^\mu\leq|f'(u)|^\mu+|f'(v)|^\mu$$

Using Hölder integral inequality, we have

$$\left|\frac{u\{u+\eta(v,u)\}}{\eta(v,u)}\int_u^{u+\eta(v,u)}\frac{f(z)}{z^2}dz-f\left(\frac{2u\{u+\eta(v,u)\}}{u+(u+\eta(v,u))}\right)\right|$$
$$\leq \quad u\eta(v,u)\{u+\eta(v,u)\}\left(\int_0^{\frac{1}{2}}t^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{1}{(\bar{A}_t)^{2\mu}}(|f'(u)|^\mu+|f'(v)|^\mu)dt\right)^{\frac{1}{\mu}}$$
$$+\left(\int_{\frac{1}{2}}^1(1-t)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^1\frac{1}{(\bar{A}_t)^{2\mu}}(|f'(u)|^\mu+|f'(v)|^\mu)dt\right)^{\frac{1}{\mu}}$$
$$= \quad u\eta(v,u)\{u+\eta(v,u)\}\left(\int_0^{\frac{1}{2}}t^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_0^{\frac{1}{2}}\frac{1}{(\bar{A}_t)^{2\mu}}dt|f'(u)|^\mu+\int_0^{\frac{1}{2}}\frac{1}{(\bar{A}_t)^{2\mu}}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}$$
$$+\left(\int_{\frac{1}{2}}^1(1-t)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_{\frac{1}{2}}^1\frac{1}{(\bar{A}_t)^{2\mu}}dt|f'(u)|^\mu+\int_{\frac{1}{2}}^1\frac{1}{(\bar{A}_t)^{2\mu}}dt|f'(v)|^\mu\right)^{\frac{1}{\mu}}$$

$$\leq \quad u\eta(v,u)\{u+\eta(v,u)\}\left(\frac{1}{2^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma}}$$
$$[\{s_1^{**}(\lambda,a,b,0,\mu)+s_2^{**}(\lambda,a,b,0,\mu)\}\left(|f'(u)|^{\mu}+|f'(v)|^{\mu}\right)]^{\frac{1}{\mu}}.$$

□

Theorem 15 *Assuming that $f : M = [u, u+\eta(v,u)] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is differentiable on the interior, M^o of M where $f' \in L_1[u, u+\eta(v,u)]$ for $u, u+\eta(v,u) \in M$ with $u < u+\eta(v,u)$, $\beta \in (0,1]$ and $\lambda, \alpha \in [0,1]$. If $|f'|$ is harmonic P-preinvex on M, we have*

(a) *If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, then*

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}[\{\{k_7^{**}(\lambda,\alpha,\beta,u,v,0)+k_8^{**}(\lambda,\alpha,\beta,u,v,0)\}$$
$$+\{k_{11}^{**}(\lambda,\alpha,\beta,u,v,0)+k_{12}^{**}(\lambda,\alpha,\beta,u,v,0)\}\}(|f'(u)|+|f'(v)|)].$$

(b) *If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, then*

$$|\Psi_f(\lambda,\beta,\alpha,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}[\{\{k_7^{**}(\lambda,\alpha,\beta,u,v,0)+k_8^{**}(\lambda,\alpha,\beta,u,v,0)\}$$
$$+k_{10}^{**}(\lambda,\alpha,\beta,u,v,0)\}(|f'(u)|+|f'(v)|)].$$

(c) *If $1-\alpha \leq (\alpha\lambda)^{\frac{1}{\beta}} \leq 1+(\lambda(\alpha-1))^{\frac{1}{\beta}}$, then*

$$|\Psi_f(\lambda,\beta,\alpha,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}[k_9^{**}(\lambda,\alpha,\beta,u,v,0)+\{k_{11}^{**}(\lambda,\alpha,\beta,u,v,0)$$
$$+k_{12}^{**}(\lambda,\alpha,\beta,u,v,0)\}(|f'(u)|+|f'(v))].$$

where $k_7^{**}(\lambda,\alpha,\beta,u,v,0)$ to $k_{12}^{**}(\lambda,\alpha,\beta,u,v,0)$ are defined in Theorem 13.

Proof. By using Lemma 2, we have

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))| \leq u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right.$$
$$\left.+\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\left|f'\left(\frac{u\{u+\eta(v,u)\}}{\bar{A}_t}\right)\right|dt\right]$$

Since $|f'|$ be harmonic P-preinvex function on the interval $[u, u+\eta(v,u)]$ for $t \in [0,1]$

$$\left|f'\left(\frac{u\{u+\eta(v,u)\}}{t(u+\eta(v,u))+(1-t)u}\right)\right| \leq |f'(u)|+|f'(v)|$$

$$|\Psi_f(\lambda,\alpha,\beta,u,u+\eta(v,u))|$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}(|f'(u)|+|f'(v)|)dt\right.$$
$$\left.+\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}(|f'(u)|+|f'(v)|)dt\right]$$
$$\leq u\eta(v,u)\{u+\eta(v,u)\}\left[\left\{\int_0^{1-\alpha}\frac{|t^{\beta}-\alpha\lambda|}{(\bar{A}_t)^2}+\int_{1-\alpha}^{1}\frac{|(t-1)^{\beta}-\lambda(\alpha-1)|}{(\bar{A}_t)^2}\right\}\right.$$
$$\left.\times(|f'(u)|+|f'(v)|)\right] \tag{3.73}$$

(a) (i) If $(\alpha\lambda)^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\begin{aligned}
&\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} dt \\
= &\int_0^{(\alpha\lambda)^{\frac{1}{\beta}}} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} dt + \int_{(\alpha\lambda)^{\frac{1}{\beta}}}^{1-\alpha} \frac{t^\beta - \alpha\lambda}{(\bar{A}_t)^2} dt \\
= &k_7^{**}(\lambda,\alpha,\beta,u,v,0) + k_8^{**}(\lambda,\alpha,\beta,u,v,0) \qquad (3.74)
\end{aligned}$$

(ii) If $(\alpha\lambda)^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\begin{aligned}
&\int_0^{1-\alpha} \frac{|t^\beta - \alpha\lambda|}{(\bar{A}_t)^2} dt \\
= &\int_0^{1-\alpha} \frac{-(t^\beta - \alpha\lambda)}{(\bar{A}_t)^2} dt \\
= &k_9^{**}(\lambda,\alpha,\beta,u,v,0) \qquad (3.75)
\end{aligned}$$

(b) (i) If $1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \leq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^{1} \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2} dt \\
= &\int_{1-\alpha}^{1} \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2} dt \\
= &k_{10}^{**}(\lambda,\alpha,\beta,u,v,0) \qquad (3.76)
\end{aligned}$$

(ii) If $1+(\lambda(\alpha-1))^{\frac{1}{\beta}} \geq 1-\alpha$, then

$$\begin{aligned}
&\int_{1-\alpha}^{1} \frac{|(t-1)^\beta - \lambda(\alpha-1)|}{(\bar{A}_t)^2} dt \\
= &\int_{1-\alpha}^{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}} \frac{-((t-1)^\beta - \lambda(\alpha-1))}{(\bar{A}_t)^2} dt \\
&+ \int_{1+(\lambda(\alpha-1))^{\frac{1}{\beta}}}^{1} \frac{(t-1)^\beta - \lambda(\alpha-1)}{(\bar{A}_t)^2} dt \\
= &k_{11}^{**}(\lambda,\alpha,\beta,u,v,0) + k_{12}^{**}(\lambda,\alpha,\beta,u,v,0) \qquad (3.77)
\end{aligned}$$

By substituting (3.74) to (3.77) in equation (3.73) gives the required result. □

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